Set Valued PDEs and Games

Melih İşeri

Joint work w/

Tianfeng Zhang

ongoing work w/

Nizar Touzi , Tianfeng Zhang




Set Valued Frameworks

Geometric Surface Evolutions

Stochastic Viability & Target Problems

Multivariate Dynamic Risk Measures

[8] FEINSTEIN, Z.; RUDLOFF, B. (2015). *Multi-portfolio time consistency for set-valued convex and coherent risk measures*. Finance and Stochastics, **19**, 67-107.

[1] ARARAT, C.; MA, J.; WU, W. (2023). *Set-valued backward stochastic differential equations*. Annals of Applied Probability, **33**(5), 3418-3448.

N-player Games

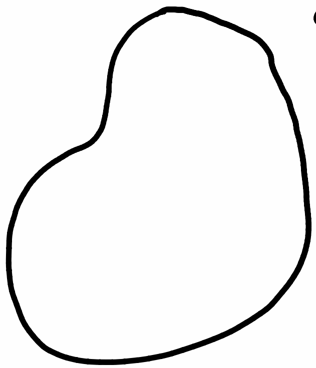
[10] FEINSTEIN, Z.; RUDLOFF, B.; ZHANG, J. (2022). *Dynamic set values for nonzero sum games with multiple equilibria*. Mathematics of Operations Research, **47**, 616-642.

Mean-field Games

[3] Melih İşeri and Jianfeng Zhang, *Set Values for Mean Field Games*, Transactions of the American Mathematical Society, **377** (2024), 7117–7174.

Multivariate Control Problems

Set Valued Calculus

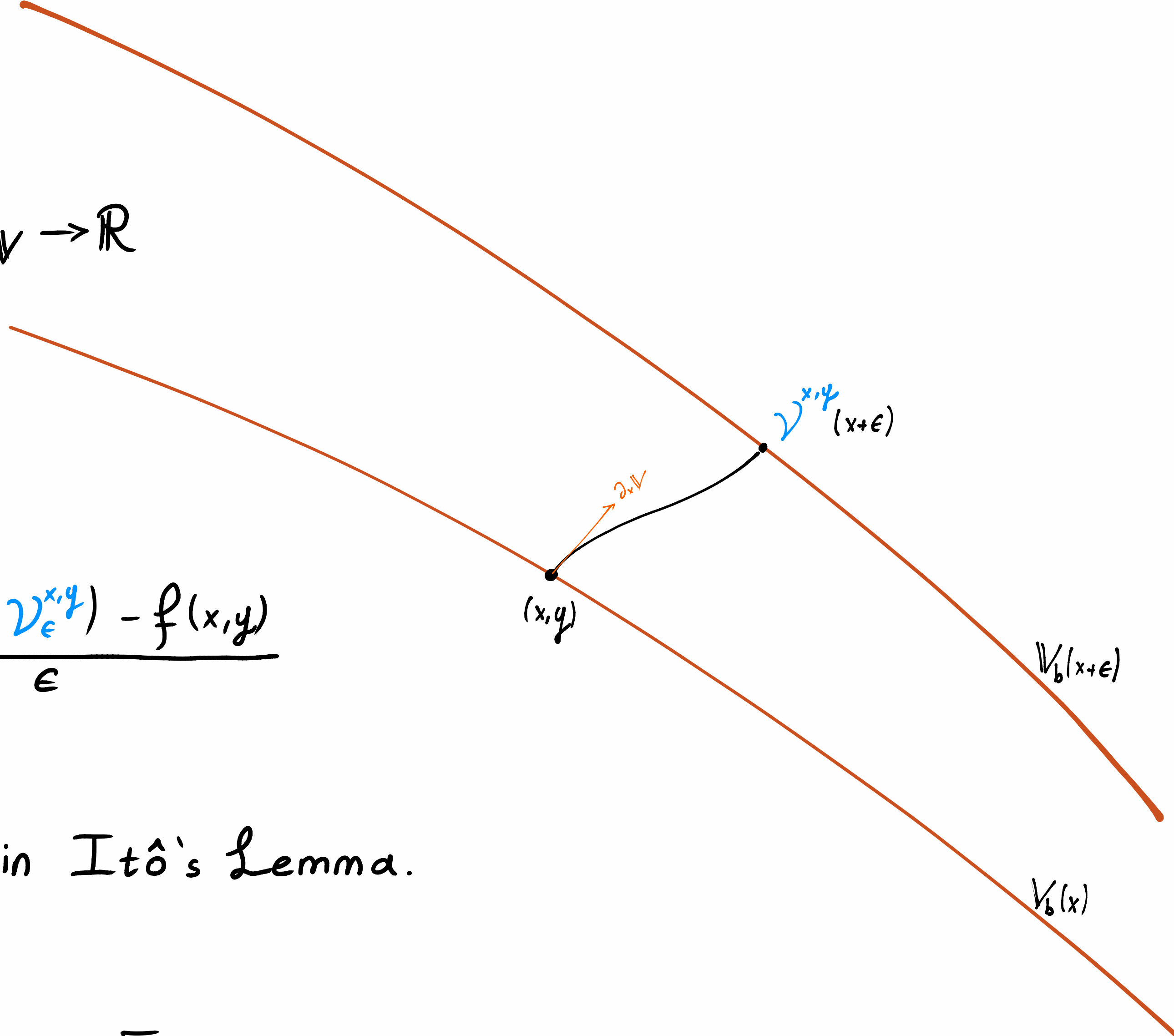
$\mathbb{V}(x) : \mathbb{R}^d \longrightarrow$  $\subset \mathbb{R}^m$

- closed, smooth boundary
- $\mathbb{V}_b(x)$ is the boundary of $\mathbb{V}(x)$

- $\mathcal{G}_{\mathbb{V}} \doteq \left\{ (x,y) : x \in \mathbb{R}^d ,\ y \in \mathbb{V}_b(x) \right\}$

- $n_{\mathbb{V}}(x,y) \doteq \mathcal{G}_{\mathbb{V}} \longrightarrow \mathbb{R}^m$ is the outward unit normal vector.

Intrinsic Derivative $f(x,y) : \mathcal{G}_{\mathbb{V}} \to \mathbb{R}$



Defn $\quad \partial_x f(x,y) \doteq \lim\limits_{\epsilon \to 0} \dfrac{f(x+\epsilon, \mathcal{V}_\epsilon^{x,y}) - f(x,y)}{\epsilon}$

Ex $\quad \partial_x n$ plays a role in Itô's Lemma.

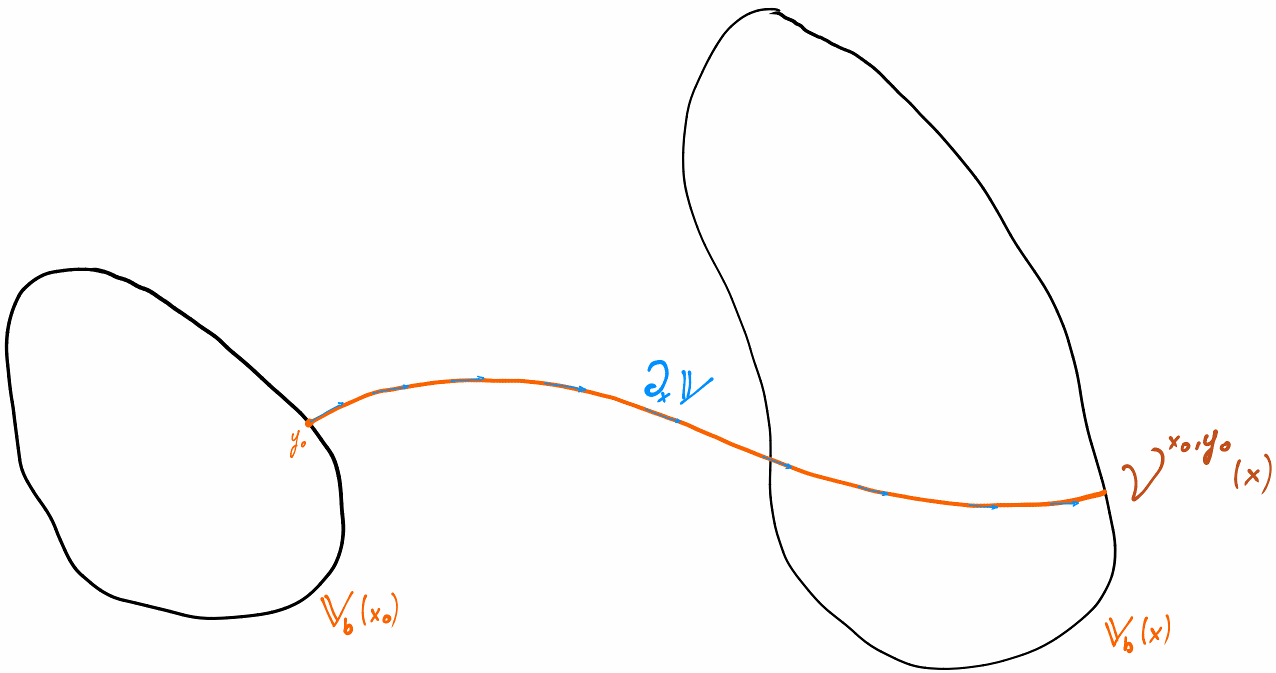
Defn $\quad \partial_x \mathbb{V}(x,y) \doteq \partial_x(y) \qquad \ulcorner f(x,y) \doteq y \lrcorner$

Defn $\quad \partial_{xx} \mathbb{V}(x,y) \doteq \partial_x(\partial_x \mathbb{V}(x,y))$

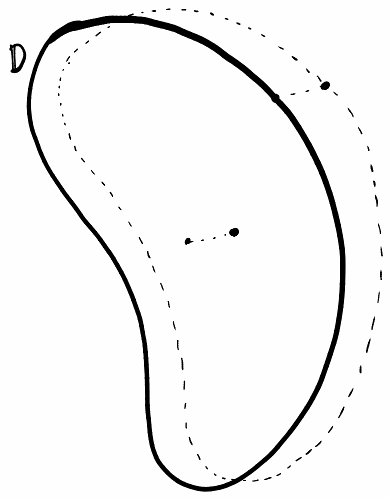
Fundamental Theorem

$$\text{“} \quad \mathbb{V}_b(x) = \mathbb{V}_b(x_0) + \int_{x_0}^{x} \partial_x \mathbb{V}(\tilde{x}, \mathbb{V}_b(\tilde{x}))\, d\tilde{x} \quad \text{”}$$

$$\mathbb{V}_b(x) = \left\{ \mathcal{V}^{x_0, y_0}(x) \; : \; \forall\, y_0 \in \mathbb{V}_b(x_0) \right\} \quad \text{where} \quad \mathcal{V}^{x_0, y_0}(x) = x_0 + \int_{x_0}^{x} \partial_x \mathbb{V}(\tilde{x}, \mathcal{V}^{x_0, y_0}(\tilde{x}))\, d\tilde{x}$$



Example 1



$$\mathbb{V}(x) = a(x) + \mathbb{D}$$

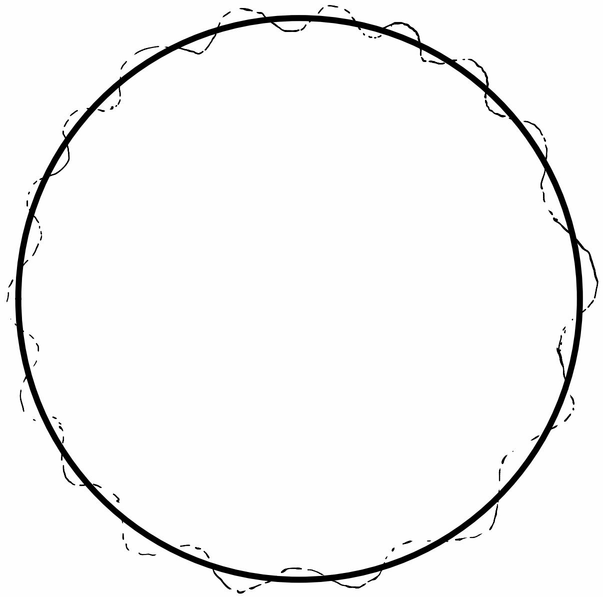
$$\partial_x \mathbb{V}(x,y) = n n^T \nabla_x a(x)$$

Example 2

$$\mathbb{V}(x) = B\left(a(x), R(x)\right)$$

$$\partial_x \mathbb{V} = n n^T \nabla_x a + n \nabla_x R$$

Example 3



$$\mathbb{V}_b(x) = \left\{ \left[1 + x\cos\left(1/x + m\theta\right)\right] (\cos\theta, \sin\theta) : \forall\, \theta \right\}$$

- continuous
- NOT Differentiable
- $n_{\mathbb{V}}$ is not continuous

Surface Evolution Equations

$$\partial_t \mathbb{V}(t,y) = h(t, y, n_{\mathbb{V}}, \partial_y n_{\mathbb{V}})$$

[18] SETHIAN, J. A. (1985). *Curvature and the evolution of fronts*. Communications in Mathematical Physics, **101**, 487-499.

[7] EVANS, L. C.; SPRUCK, J. (1991). *Motion of level sets by mean curvature. I*. Journal of Differential Geometry, **33**, 635-681.

[4] BARLES, G.; SONER, H. M.; SOUGANIDIS, P. E. (1993). *Front propagation and phase field theory*. SIAM Journal on Control and Optimization, **31**(2), 439-469.

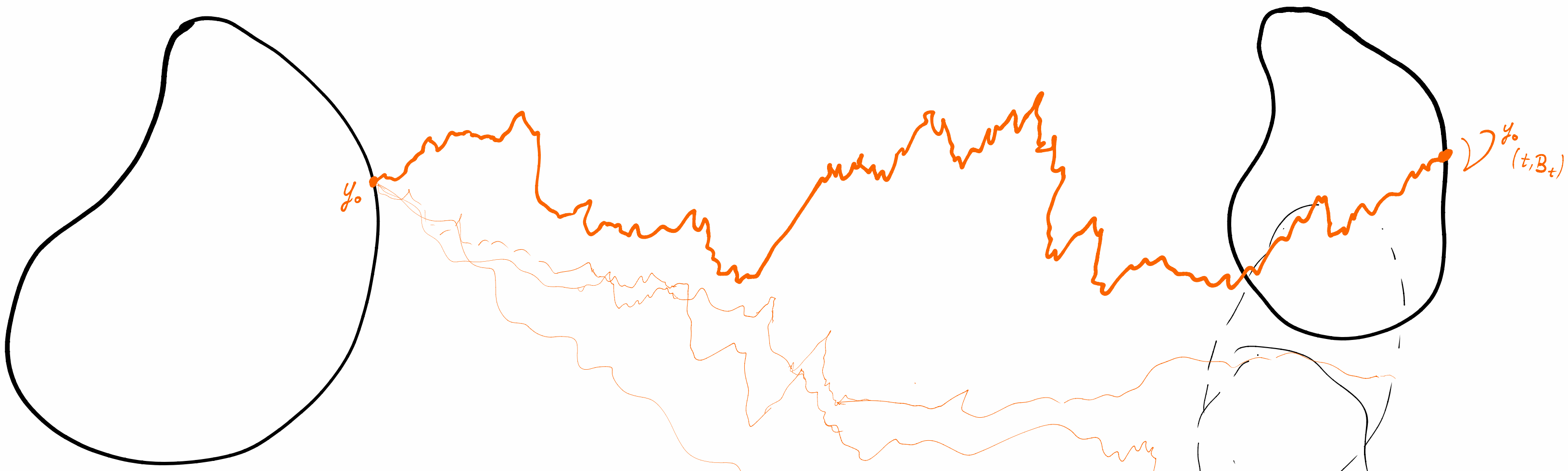
[19] SONER, H. M. (1993). *Motion of a set by the curvature of its boundary*. Journal of Differential Equations, **101**, 313-372.

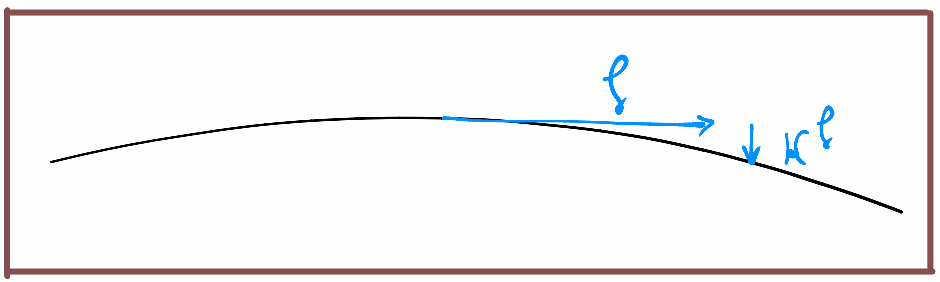
[22] SONER, H. M.; TOUZI, N. (2003). *A stochastic representation for mean curvature type geometric flows*. The Annals of Probability, Vol. 31 No. 3, 1145-1165.

[11] GIGA, Y. (2006). *Surface evolution equations: A level set approach*. Monographs in Mathematics, Birkhäuser Basel.

Itô's Formula

$$\mathbb{V}_b(t, B_t) = \mathbb{V}_b(0,0) + \int_0^t \left[\partial_t \mathbb{V} + \frac{1}{2} \partial_{xx} \mathbb{V} + \kappa^{\ell} \right] ds + \int_0^t \left[\partial_x \mathbb{V} + \ell \right] dB_s$$



Multivariate Control Problem

Dynamics

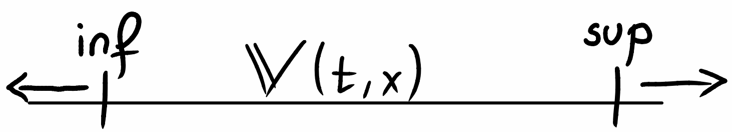
$$X_s^{t,x,\alpha} = x + \int_t^s b(r, X_r^{t,x,\alpha}, \alpha_r)\, dr + \int_t^s \sigma(r, X_r^{t,x,\alpha}, \alpha_r)\, dB_r$$

Value

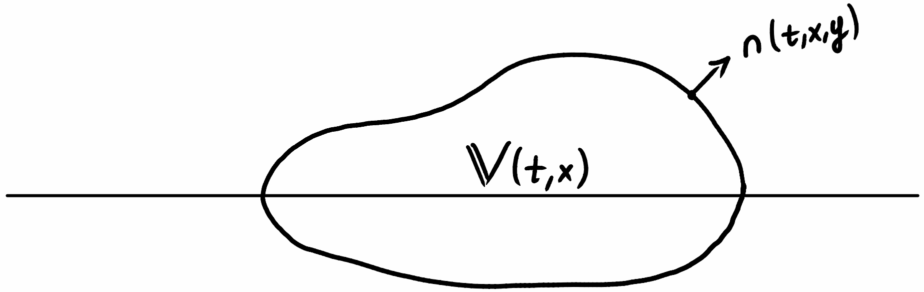
$$Y_s^{t,x,\alpha} = g(X_T^{t,x,\alpha}) + \int_s^T f(r, X_r^{t,x,\alpha}, \alpha_r, Y_r^{t,x,\alpha}, Z_r^{t,x,\alpha}) - \int_s^T Z_r^{t,x,\alpha}\, dB_r$$

Set Value: $\mathbb{V}(t,x) = \{ Y_t^{t,x,\alpha} : \forall \alpha \in \mathcal{A} \}$

In one dimension



In higher dimensions



Motivation : Games, Portfolio of assets, Target Problems, Risk of Many Institutions

Application : Time inconsistent problems

[15] KARNAM, C.; MA, J.; ZHANG, J. (2017). *Dynamic Approaches for Some Time Inconsistent Optimization Problems*. Annals of Applied Probability, **27**, 3435-3477.

Mean-Variance :

$$\sup_\alpha \mathbb{E}\left[X_T^{t,x,\alpha}\right] - \frac{\lambda}{2}\left(\mathbb{E}\left[|X_T^{t,x,\alpha}|^2\right] - \mathbb{E}\left[X_T^{t,x,\alpha}\right]^2\right) = \sup_{y \in \mathbb{V}(t,x)} \varphi(y)$$

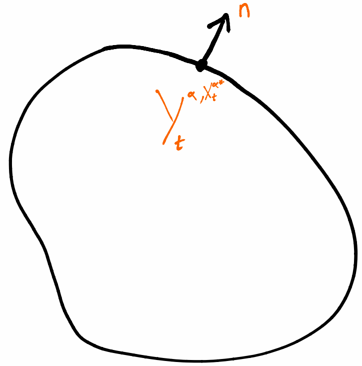
$$\ulcorner\, Y^1 = X_T - \int^T Z^1 dB, \quad Y^2 = |X_T|^2 - \int^T Z^2 dB, \quad \varphi(y_1, y_2) = y_1 - \frac{\lambda}{2} y_1 + \frac{\lambda}{2} y_2^2 \,\lrcorner$$

Moving Scalarization $(\varphi(y) \doteq \lambda \cdot y)$

$$\sup_\alpha Y_0^\alpha \cdot \lambda \longrightarrow \alpha^* \text{ optimal}$$

$$\sup_\alpha Y_t^{\alpha, X_t^{\alpha^*}} \cdot \lambda$$

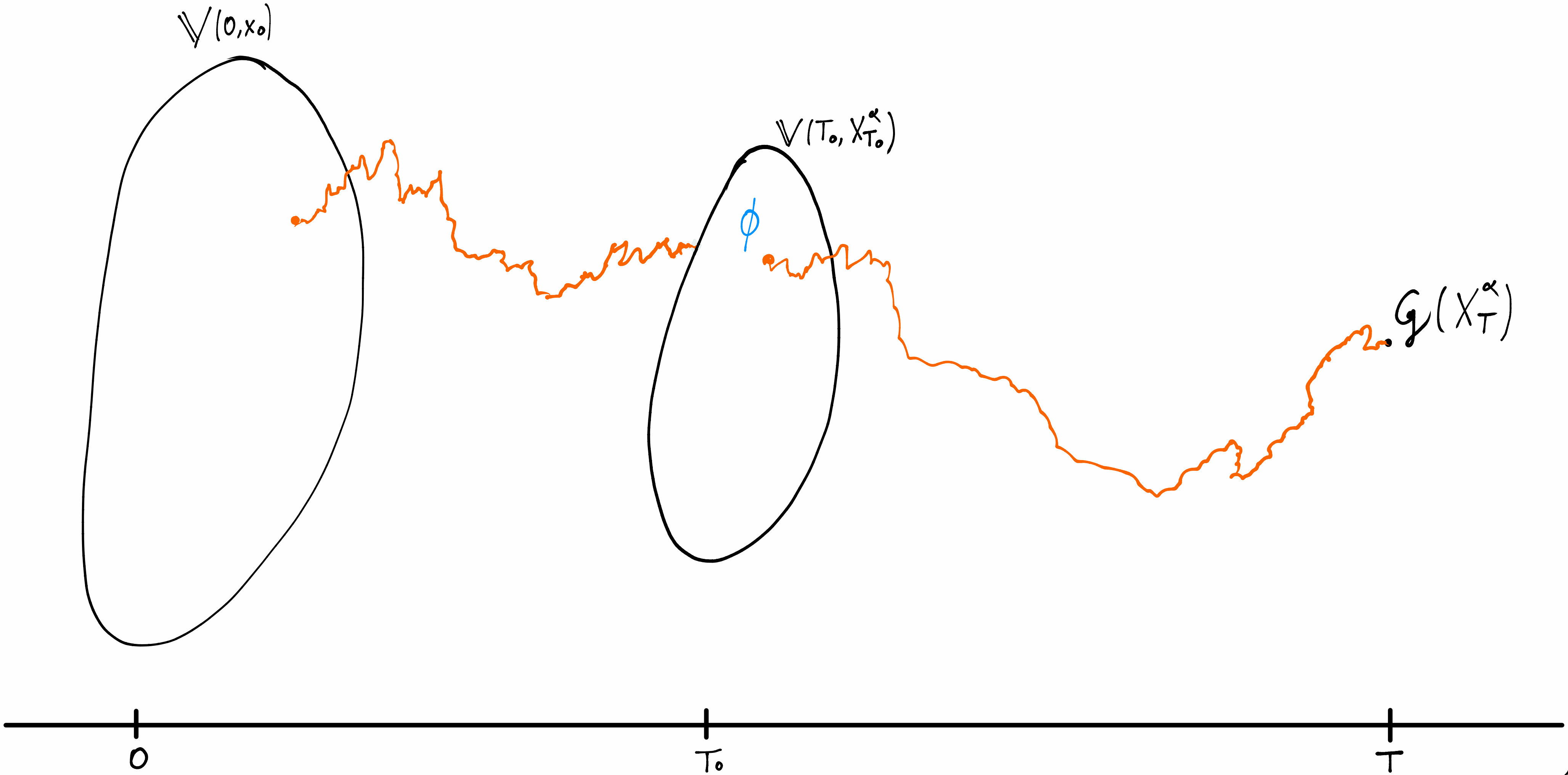
$$\sup_\alpha Y_t^{\alpha, X_t^{\alpha^*}} \cdot n\left(t, X_t^{\alpha^*}, Y_t^{\alpha, X_t^{\alpha^*}}\right)$$



[9] FEINSTEIN, Z.; RUDLOFF, B. (2022). *Time consistency for scalar multivariate risk measures*. Statistics & Risk Modeling, **38**, 71-90.

Time - Consistency

$$\mathbb{V}(0,x_0) = \left\{ Y_0^{T_0,\phi;\alpha} \; : \; \forall \alpha \in \mathcal{A} \; , \; \phi \in \mathbb{V}(T_0, X_{T_0}^{\alpha}) \right\}$$



$\mathcal{HJB}$ $$\sup_{a,\xi} n^T\left[\partial_t \mathbb{V} + \frac{1}{2}\partial_{xx}\mathbb{V} + K^{\xi} + f\right](t,x,y,a) = 0$$

$$\forall (t,x,y) \in \mathcal{G}_{\mathbb{V}} \quad \text{and} \quad \mathbb{V}(T,x) = g(x)$$

Main Theorem

(i) Suppose $\mathbb{V} \in C^{1,2}$. Then $\mathbb{V}$ is a classical solution of $\mathcal{HJB}$.

(ii) Suppose $\mathbb{U} \in C^{1,2}$ is a classical solution to $\mathcal{HJB}$. Then $\mathbb{V} = \mathbb{U}$.

(ii') Furthermore, if there exists optimal arguments I^* and ξ^*, then

for any $y \in \mathbb{V}_b(t,x)$, $\exists$ optimal α^* : $y = Y_t^{t,x,\alpha^*}$ and $Y_s^{t,x,\alpha^*} \in \mathbb{V}_b(s, X_s^*)$

Moving Scalarization: Given λ, choose y^λ solving $\sup_y \lambda \cdot y$. By verification, construct α^*, X^*, Y^*, and since $Y_t^* \in \mathbb{V}_b(t, X_t^*)$, α^* is optimal for $n(t, X_t^*, Y_t^*)$.

Mean-Variance: Introduce $\psi(y_1, y_2) = (y_1, y_2 - y_1^2)$ and $\tilde{\mathbb{V}}(t,x) \doteq \left\{ \psi(y) : \forall y \in \mathbb{V}(t,x) \right\}$

Then, $\sup_{y \in \mathbb{V}(t,x)} \varphi(y) = \sup_{\tilde{y} \in \tilde{\mathbb{V}}(t,x)} \tilde{y}^1 - \frac{\lambda}{2} \tilde{y}^2$ which is linear! Then, dynamic mean-variance

$$V_t \doteq \operatorname{ess\,sup}_\alpha \left\{ \mathbb{E}\left[X_T^\alpha \mid \mathcal{F}_t \right] - \frac{\Lambda(t, X^*_{[0,t]})}{2} \operatorname{Var}\left(X_T^\alpha \mid \mathcal{F}_t \right) \right\}$$ is time-consistent.

$$\alpha^* = -X_t^* + x_0 + \frac{1}{\lambda} e^T, \quad \Lambda(t, x_{[0,t]}) = \frac{\lambda e^{T-t}}{e^T - \lambda(x_t - x_0)}$$

$$V_t = \frac{1}{2}\left(1 + e^{-(T-t)}\right) X_t^* + \frac{1}{2}\left(1 - e^{-(T-t)}\right) x_0 + \frac{e^T}{2\lambda}\left(1 - e^{-(T-t)}\right)$$

Set Values & Set Hamiltonian of Games

$$Y_s^{t,x,\vec{\alpha},i} = g^i(X_T^{t,x}) + \int_s^T h^i(r, X_r^{t,x}, Z_r^{t,x,\vec{\alpha},i}, \vec{\alpha}_r)\,dr - \int_s^T Z_r^{t,x,\vec{\alpha},i}\,dB_r$$

$$h^i(t,x,z,\vec{a}) := f^i(t,x,\vec{a}) + z\,b(t,x,\vec{a})$$

Global

$$J^i(t,x,\vec{\alpha}) \geq J^i(t,x,(\vec{\alpha}^{-i},\alpha^i))$$

$$\mathbb{V}(t,x) := \{ \vec{J}(t,x,\vec{\alpha}^*) : \forall\, \alpha^* \in \mathcal{E}(t,x) \}$$

Local

$$h^i(t,x,z,\vec{a}) \geq h^i(t,x,z,(\vec{a}^{-i},a^i))$$

$$\mathbb{H}(t,x,z) := \{ \vec{h}(t,x,z,\vec{a}^*) : \forall\, \vec{a}^* \in \mathcal{E}(t,x,z) \}$$

- Assumption : $\mathbb{H}(t,x,z)$ non-empty & continuous

Seperabality

Defn $\mathbb{H}$ is L-seperable : $\mathbb{H}(t,x,z) = cl\{H^n(t,x,z) : \forall n\}$ where H^n measurable in (t,x), L-Lipschitz in z.

Theorem

$$\mathbb{V}(t,x) = cl\{\, Y^{I}_{t,x} : I \in \mathcal{I}_t \}$$

$$Y_s^{t,x,I,i} = g^i(X_T^{t,x}) + \int_s^T H^{I,i}(r, X_\cdot^{t,x}, Z_r^{t,x,I,i})\,dr - \int_s^T Z_r^{t,x,I,i}\,dB_r$$

$$H^{I}(t,x_\cdot,z) \doteq H^{I(t,x_\cdot)}(t,x_t,z)$$

[2] Bixing Qiao and Jianfeng Zhang, *Set Values of Dynamic Nonzero Sum Games and Set Valued Hamiltonians*, arXiv preprint arXiv:2408.09047 [math.OC], 2024. Available at https://arxiv.org/abs/2408.09047.

$$\text{PDE} \quad \sup_{\vec{a} \in \mathcal{E}(\cdot, \partial_x \mathbb{V} + \zeta^*)} n^T \left[\partial_t \mathbb{V}(\cdot) + h(\cdot, \partial_x \mathbb{V}, a) + \frac{1}{2} tr(\partial_{xx} \mathbb{V}) - \mathcal{K}^{\zeta^*}(\cdot) \right](t,x,y) = 0$$

Theorem

(i) Suppose $\mathbb{V} \in C^{1,2}$. Then $\mathbb{V}$ is a classical solution to PDE

(ii) Suppose $\mathbb{U} \in C^{1,2}$ and is a classical solution to PDE. Then $\mathbb{U} = \mathbb{V}$.

(ii') Furthermore, if there exists $\mathcal{I}^*(t,x,y) \in \mathcal{E}(\cdot, \partial_x \mathbb{V} + \zeta^*)$ an optimal argument,

then $\alpha^*_t = \mathcal{I}^*(t, X_t, \Upsilon^*)$ and $Y^{\alpha^*}_t = \Upsilon^*_t \in \mathbb{V}_b(t, X_t)$. Moreover, $n(t, X_t, \Upsilon^*_t)$ is absolutely continuous.

Geometric Properties

Theorem $\mathbb{V}(t,x)$ is a compact, convex set.

Theorem Fix $I_0 \equiv 0$ and take any subspace S of $\mathbb{R}^N$. Then,

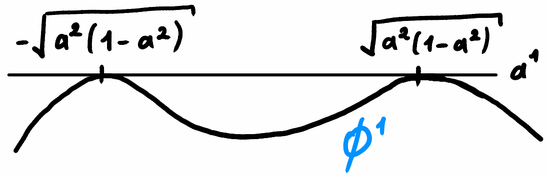
$$\left\{ y - Y_t^0 : \forall y \in \mathbb{V}(t,x) \right\} \subset S \text{ iff } \left\{ H^I(s, X_\cdot^{t,x}, Z_s^0) - H^0(s, X_\cdot^{t,x}, Z_s^0) : \forall I \in \mathcal{I}_t \right\} \subset S \quad dt \times d\mathbb{P}\text{-a.s.}$$

Corollary If there exists $(t_0, x_0) \in [t,T]$ s.t.

convex hull of $\mathbb{H}(t_0, x_0, z)$ has non-empty interior $\forall z$,

then $\mathbb{V}(t,x)$ has also non-empty interior.

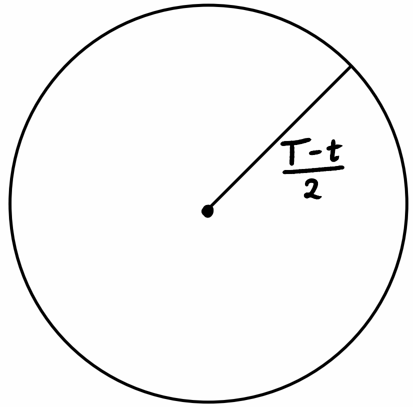
Examples



Simple Ball: $N=2,\ \mathbb{A}=[0,1],\ b\equiv 0,\ g\equiv 0,\ f(a^1,a^2) = \left(\phi^1(a^1,a^2)+a^2,\ a^1\right)$

$$\mathbb{H}(t,x,z) = \mathbb{H} = \left\{ h(a^*) : \forall a^* \in \mathcal{E} \right\} = B_b(1/2, 1/2)$$

$$\mathbb{V}(t) = cl\left\{ \mathbb{E}\left[\int_t^T H^I ds \right] : \forall I \in \mathcal{I}_t \right\}$$



Deterministic: $b(a^1,a^2) = \left(a^1(2a^2-1),0\right) = f(a^1,a^2),\ g(x^1,x^2) = \left(0,|x^1|^2\right)$

$$X_s^{t,x,\alpha} = x + \int_t^s b(\alpha_r)\,dr,\quad J(t,x,\alpha) = g(X_T^{t,x,\alpha}) + \int_t^T f(\alpha_r)\,dr$$

not convex!

$$J^1(t,x,\alpha) = \int_t^T \alpha_s^1(2\alpha^2-1)\,ds,\quad J^2(t,x,\alpha) = \left(X_T^{t,x,\alpha}\right)^2 = \left(x^1 + J^1(t,x,\alpha)\right)^2$$

Thank

You