

Set-Valued PDEs: Stochastic Control & Games

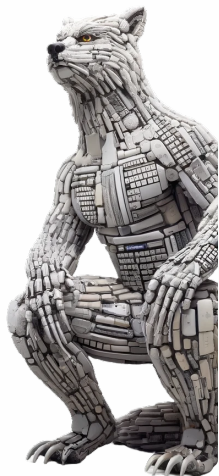
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Joint work w/

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ongoing work w/

Nizar Touzi , Tianfeng Zhang



Set Valued Frameworks

Geometric Surface Evolutions

Stochastic Viability & Target Problems

Multivariate Dynamic Risk Measures

- [8] FEINSTEIN, Z.; RUDLOFF, B. (2015). *Multi-portfolio time consistency for set-valued convex and coherent risk measures*. Finance and Stochastics, **19**, 67-107.
- [1] ARARAT, C.; MA, J.; WU, W. (2023). *Set-valued backward stochastic differential equations*. Annals of Applied Probability, **33**(5), 3418-3448.

N-player Games

- [10] FEINSTEIN, Z.; RUDLOFF, B.; ZHANG, J. (2022). *Dynamic set values for nonzero sum games with multiple equilibria*. Mathematics of Operations Research, **47**, 616-642.

Mean-field Games

- [3] Melih İşeri and Jianfeng Zhang, *Set Values for Mean Field Games*, Transactions of the American Mathematical Society, **377** (2024), 7117-7174.

Multivariate Control Problems

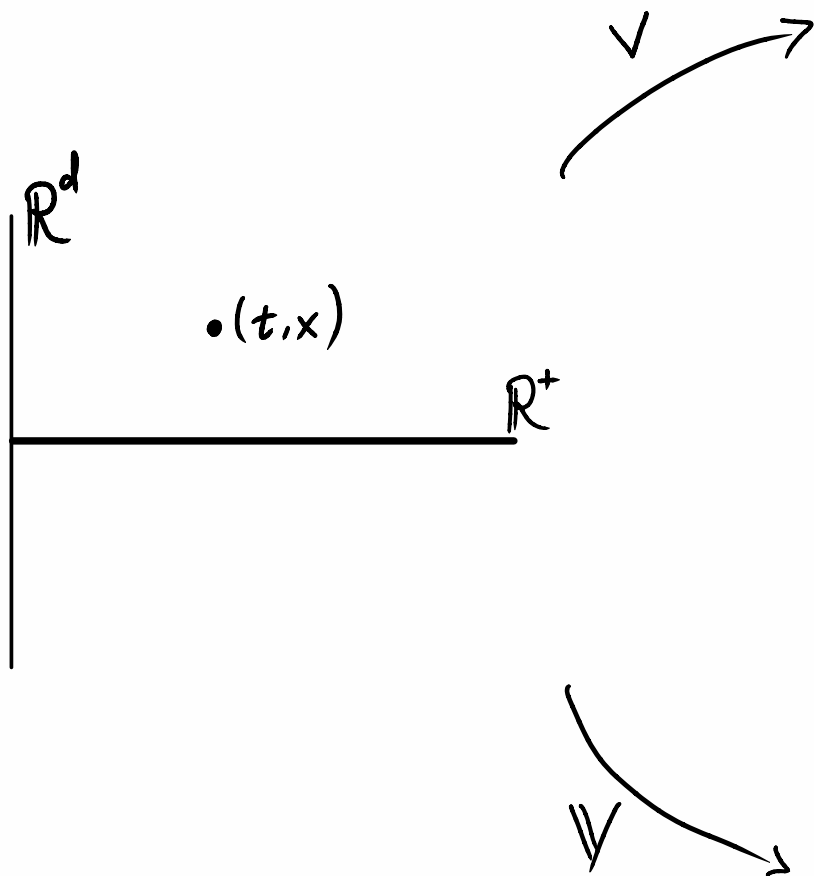
- [5] M. İşeri and J. Zhang, *Set valued Hamilton-Jacobi-Bellman equations*, Annals of Probability, to appear.

Surface Evolution Equations

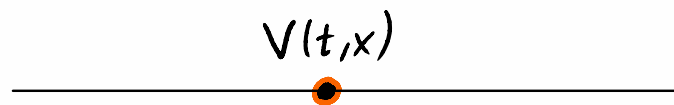
$$\partial_t V(t, y) = h(t, y, n_V, \partial_y n_V)$$

- [18] SETHIAN, J. A. (1985). *Curvature and the evolution of fronts*. Communications in Mathematical Physics, **101**, 487-499.
- [7] EVANS, L. C.; SPRUCK, J. (1991). *Motion of level sets by mean curvature. I*. Journal of Differential Geometry, **33**, 635-681.
- [4] BARLES, G.; SONER, H. M.; SOUGANIDIS, P. E. (1993). *Front propagation and phase field theory*. SIAM Journal on Control and Optimization, **31**(2), 439-469.
- [19] SONER, H. M. (1993). *Motion of a set by the curvature of its boundary*. Journal of Differential Equations, **101**, 313-372.
- [22] SONER, H. M.; TOUZI, N. (2003). *A stochastic representation for mean curvature type geometric flows*. The Annals of Probability, Vol. 31 No. 3, 1145-1165.
- [11] GIGA, Y. (2006). *Surface evolution equations: A level set approach*. Monographs in Mathematics, Birkhäuser Basel.

State Space

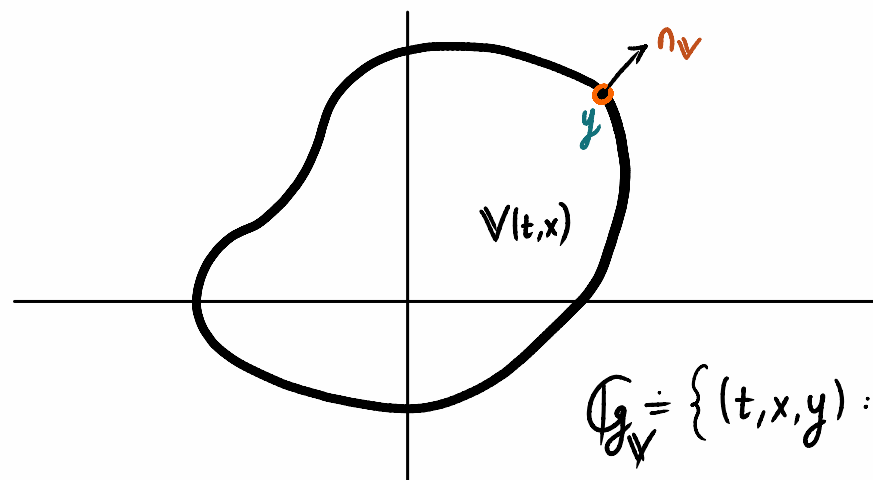


Value Space



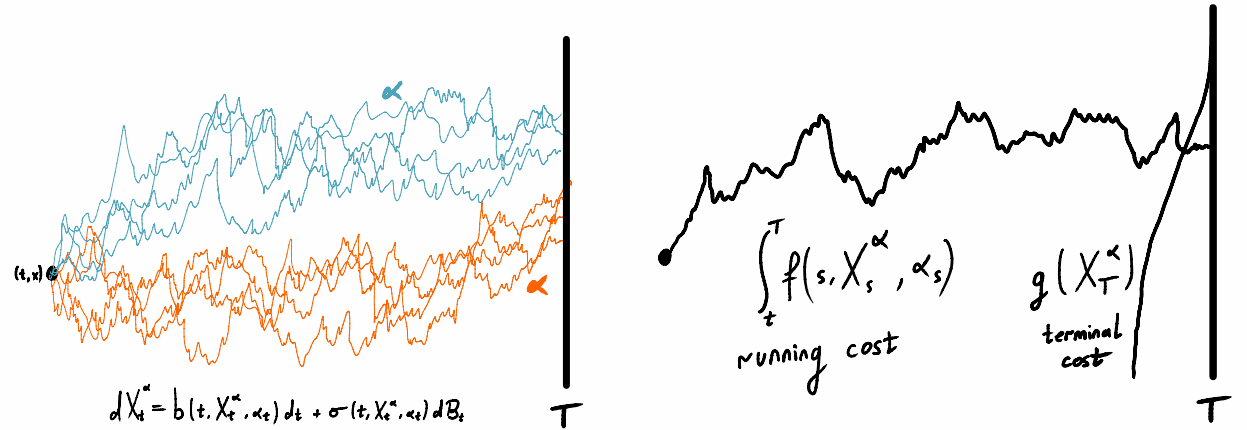
$$\partial_t v(t, x) + H(\cdot, \partial_x v, \partial_{xx} v)(t, x) = 0$$

$$n_V^T \partial_t V(t, x, y) + H(\cdot, \partial_x V, \partial_{xx} V, n_V, \partial_x n_V, \partial_y n_V)(t, x, y) = 0$$



$$\mathcal{G}_V \doteq \{(t, x, y) : y \in V(t, x)\}$$

Stochastic Control Problem



$$H(t, x, z, \gamma) \doteq \sup_a \inf \left\{ z b + \frac{1}{2} \text{tr}(\sigma \sigma^T \gamma) + f(\cdot, z \sigma, a) \right\} (t, x)$$

$$H(t, x, z, \gamma, \eta, k, N) \doteq$$

$$\sup_a \sup_{\ell} \eta^T \left\{ z b + \frac{1}{2} \text{tr}(\sigma \sigma^T \gamma) - \frac{1}{2} \text{tr}(\ell \ell^T N + 2 \ell^T k) \eta + f(\cdot, z \sigma + \ell, a) \right\} (t, x, y)$$

Proposition f has bounded, Lipschitz derivative $\Rightarrow V(t, x)$ is convex

* Quadratic growth on z / Deterministic setting yields non-convex example.

~ proof

Dual Representation :

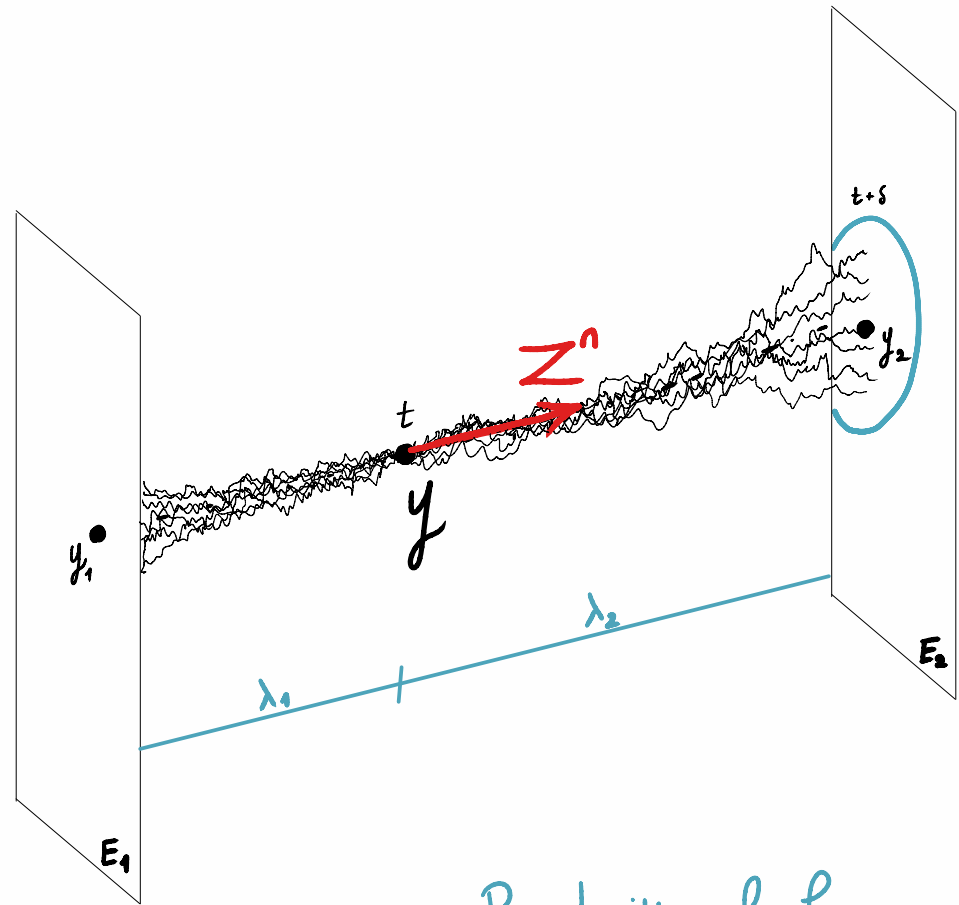
$$V(t, x) = \{y \in \mathbb{R}^m : W(t, x, y) = 0\} \quad \text{where}$$

$$W(t, x, y) = \inf_{\alpha} \inf_z E \left[\left| y - \int_t^T f + \int_t^T z dB_s - g(X_T^{t, x, \alpha}) \right|^2 \right]$$

Dynamic Programming Principle ~

$$W(t, x, y) \leq \underbrace{W(t+\delta, x, y_1)}_{\text{right-continuous}} \underbrace{\mathbb{P}(\mathcal{T} \in E_1)}_{\rightarrow \lambda_1 / \lambda_1 + \lambda_2} + W(t+\delta, x, y_2) \mathbb{P}(\mathcal{T} \in E_2) + w(\delta)$$

Regularities of f



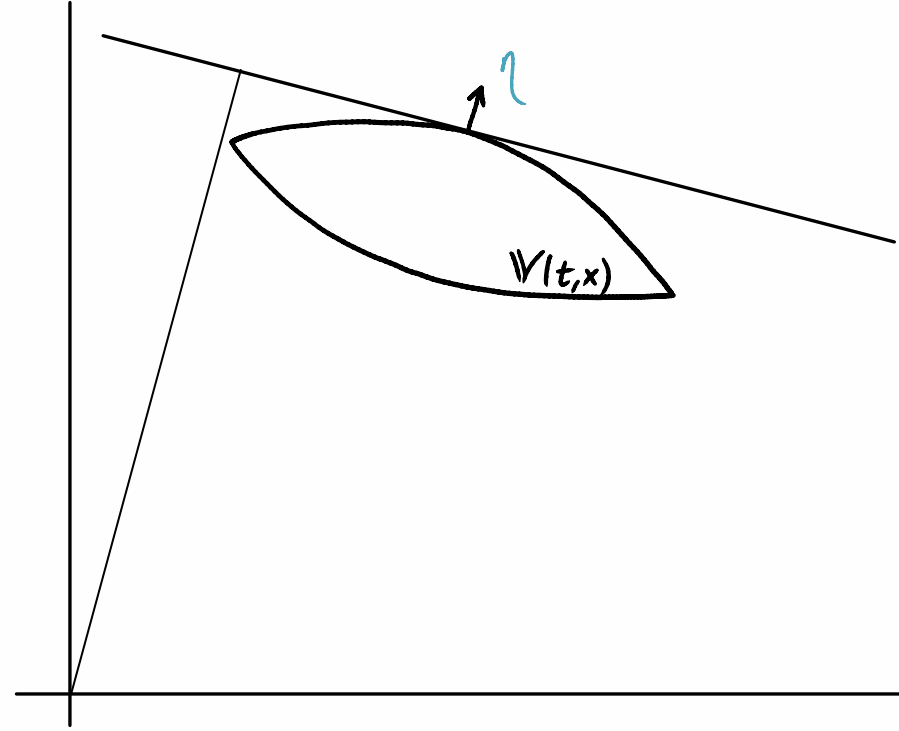
Convex: Support function

$$S_V(t, x, \eta) \doteq \sup \eta \cdot V(t, x)$$

$$\partial_t s_V + H^s(\cdot, \partial_x s_V, \partial_{xx}^2 s_V)(t, x, \eta) = 0$$

$$H^s(t, x, \eta, \partial_x s, \partial_{xx}^2 s) =$$

$$\sup_a \sup_{\ell} \inf_{\ell} \left\{ s_x^T b + \eta^T f(\cdot, \eta s_x^T \sigma + \ell) + \frac{1}{2} \text{tr} \left(\begin{pmatrix} \sigma \\ \ell \end{pmatrix} \begin{pmatrix} \sigma \\ \ell \end{pmatrix}^T \partial_{xx}^2 s \right) - \text{tr}(\sigma \sigma^T \ell^T \ell) \right\}$$



Motivation : Mean-field Games, Portfolio of assets, Target Problems, Risk of Many Institutions

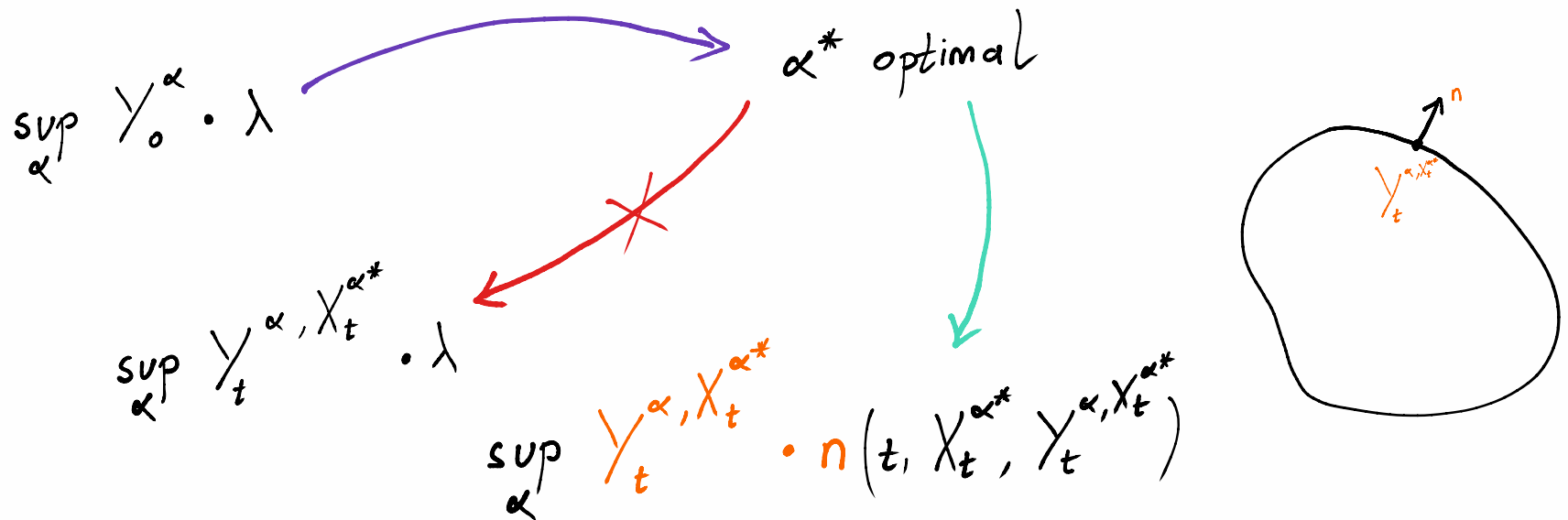
Application : Time inconsistent problems & Games - PDE under multiple equilibria

Mean-Variance

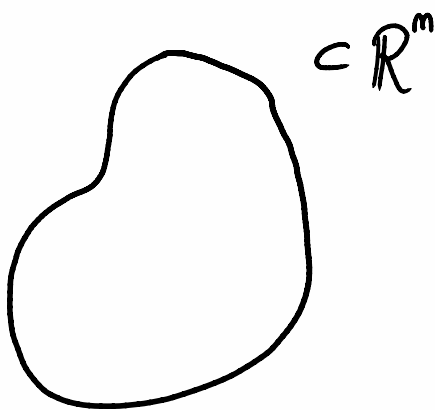
$$\sup_{\alpha} \mathbb{E}[X_T^{t,x,\alpha}] - \frac{\lambda}{2} (\mathbb{E}[|X_T^{t,x,\alpha}|^2] - |\mathbb{E}[X_T^{t,x,\alpha}]|^2) = \sup_{y \in \mathcal{V}(t,x)} \varphi(y)$$

$$\begin{cases} Y^1 = X_T - \int_t^T Z^1 dB, \\ Y^2 = |X_T|^2 - \int_t^T Z^2 dB, \end{cases} \quad \varphi(y_1, y_2) = y_1 - \frac{\lambda}{2} (y_2 - |y_1|^2)$$

Moving Scalarization $\varphi(y) = \lambda \cdot y$



Set Valued Calculus

$$V(x) : \mathbb{R}^d \rightarrow \text{set} \subset \mathbb{R}^m$$


- closed, smooth boundary
- $V_b(x)$ is the boundary of $V(x)$

- $G_V \doteq \{(x, y) : x \in \mathbb{R}^d, y \in V_b(x)\}$ "Graph of V_b "

- $n_V(x, y) : G_V \rightarrow \mathbb{R}^m$ is the outward unit normal vector.

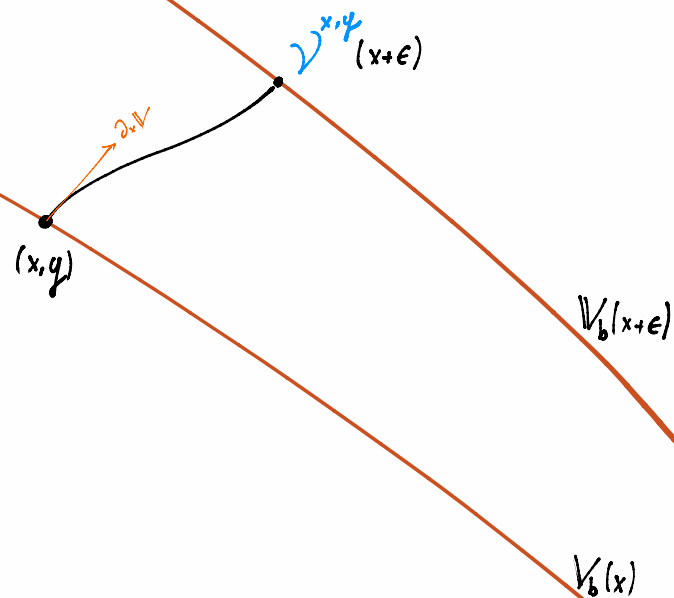
Intrinsic Derivative $f(x,y) : \mathbb{G}_V \rightarrow \mathbb{R}$

Defn $\partial_x f(x,y) \doteq \lim_{\epsilon \rightarrow 0} \frac{f(x+\epsilon, \mathcal{V}_\epsilon^{x,y}) - f(x,y)}{\epsilon}$

Ex $\partial_x n$ plays a role in Itô's Lemma.

Defn $\partial_x V(x,y) \doteq \partial_x(y) \quad \lceil f(x,y) \doteq y \rceil$

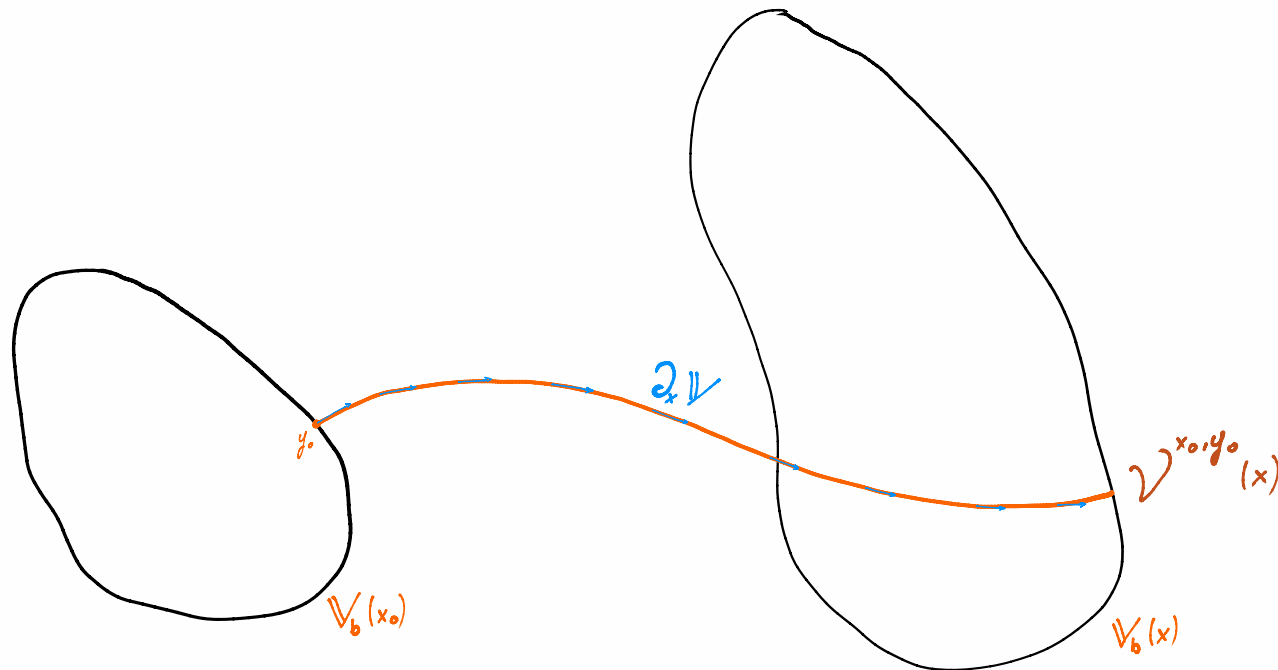
Defn $\partial_{xx} V(x,y) \doteq \partial_x(\partial_x V(x,y))$



Fundamental Theorem

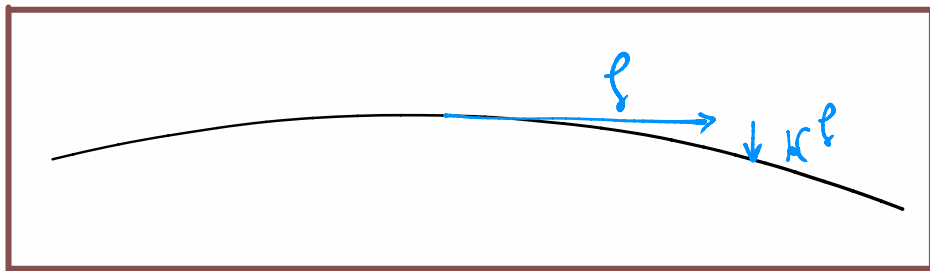
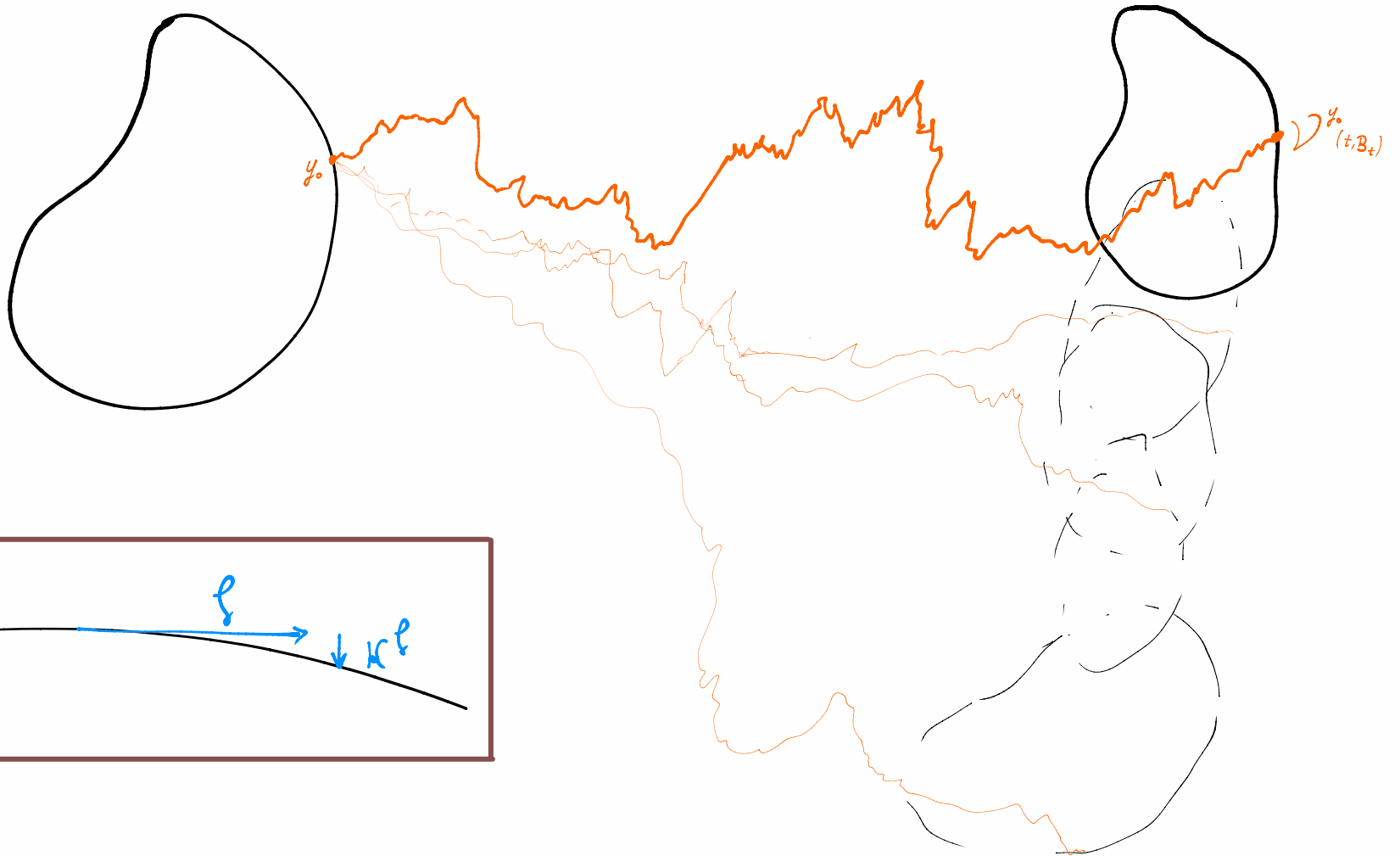
$$" \mathbb{V}_b(x) = \mathbb{V}_b(x_0) + \int_{x_0}^x \partial_x \mathbb{V}(\tilde{x}, \mathbb{V}_b(\tilde{x})) d\tilde{x} "$$

$$\mathbb{V}_b(x) = \left\{ \mathcal{V}^{x_0, y_0}_{(x)} : \forall y_0 \in \mathbb{V}_b(x_0) \right\} \quad \text{where} \quad \mathcal{V}^{x_0, y_0}_{(x)} = x_0 + \int_{x_0}^x \partial_x \mathbb{V}(\tilde{x}, \mathcal{V}^{x_0, y_0}_{(\tilde{x})}) d\tilde{x}$$



Itô's Formula

$$V_b(t, B_t) = V_b(0, 0) + \int_0^t \left[\partial_t V + \frac{1}{2} \partial_{xx} V + \kappa \ell \right] ds + \int_0^t \left[\partial_x V + \ell \right] dB_s$$



Multivariate Control Problem

Dynamics

$$X_s^{t,x,\alpha} = x + \int_t^s b(r, X_r^{t,x,\alpha}, \alpha_r) dr + \int_t^s \sigma(r, X_r^{t,x,\alpha}, \alpha_r) dB_r$$

Value

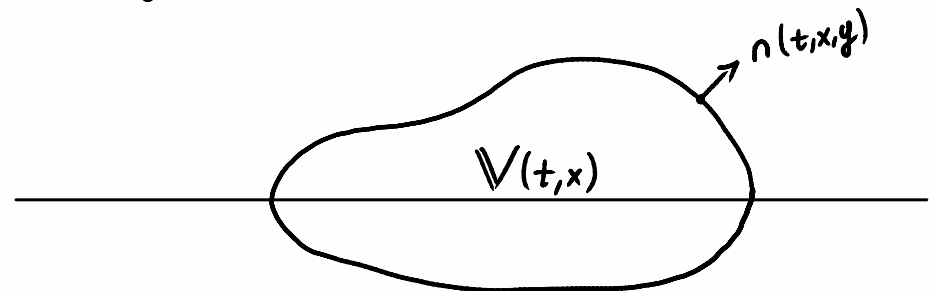
$$Y_s^{t,x,\alpha} = g(X_T^{t,x,\alpha}) + \int_s^T f(r, X_r^{t,x,\alpha}, \alpha_r, Y_r^{t,x,\alpha}, Z_r^{t,x,\alpha}) - \int_s^T Z_r^{t,x,\alpha} dB_r$$

Set Value: $V(t, x) = \{ Y_t^{t,x,\alpha} : \forall \alpha \in \mathcal{A} \}$

In one dimension

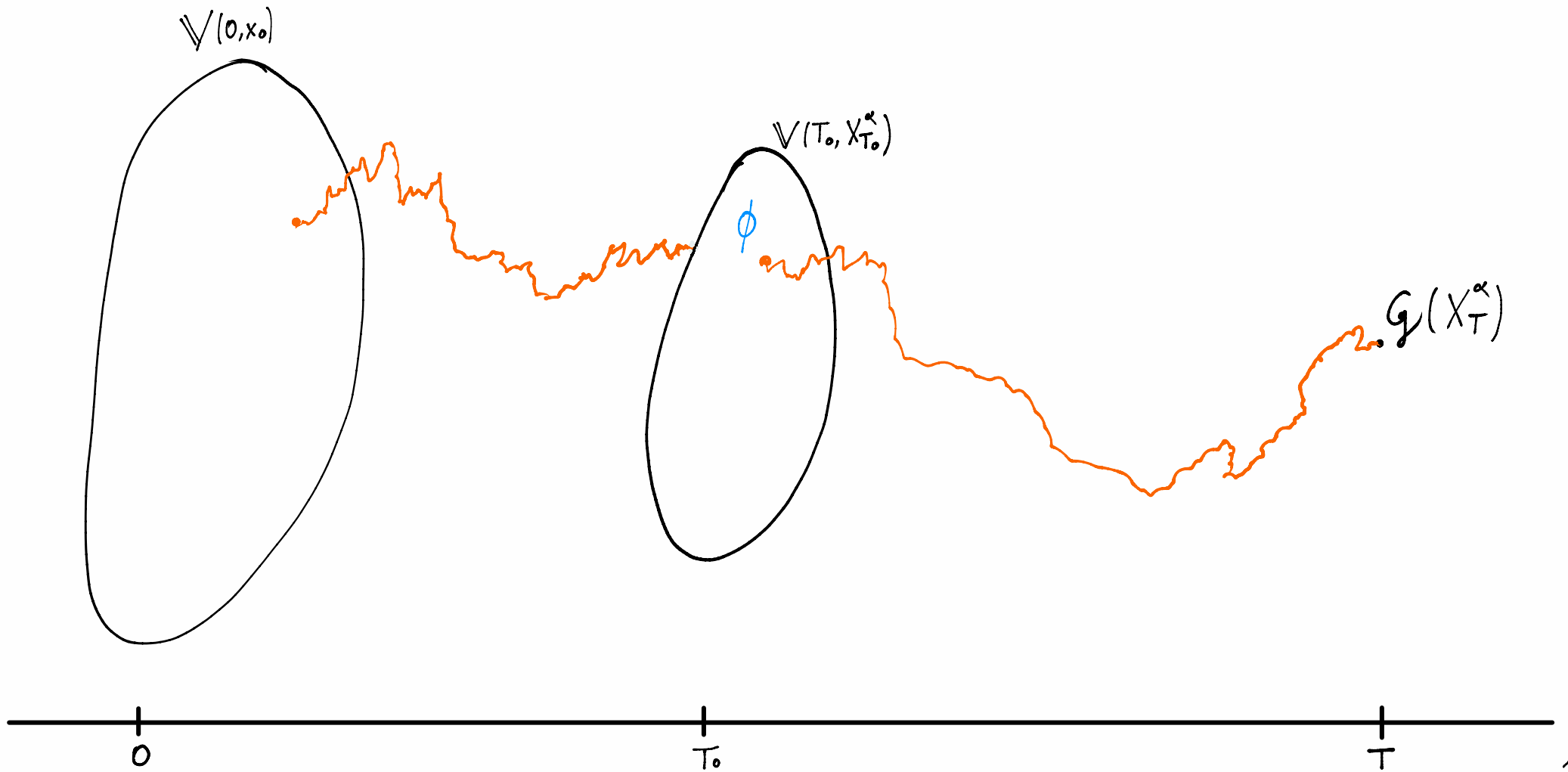
$$\leftarrow \inf \quad V(t, x) \quad \sup \rightarrow$$

In higher dimensions



Dynamic Programming Principle (Time-consistency)

$$V(0, x_0) = \left\{ V_0^{T_0, \phi, \alpha} : \forall \alpha \in \mathcal{A}, \phi \in V(T_0, X_{T_0}^\alpha) \right\}$$

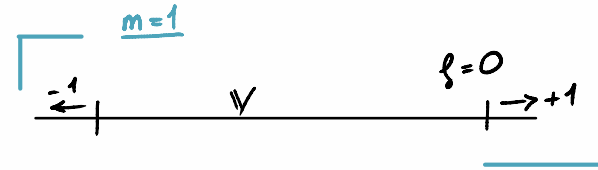


HJB

$$\sup_{\alpha, f} n^T \left[\partial_t V + \frac{1}{2} \partial_{xx} V + K^f + f \right](t, x, y, \alpha) = 0$$

$$V(t, x, y) \in \mathbb{G}_V \text{ and } V(T, x) = g(x)$$

Main Theorem



- (i) Suppose $V \in C^{1,2}$. Then V is a classical solution of **HJB**.
- (ii) Suppose $U \in C^{1,2}$ is a classical solution to **HJB**. Then $V = U$.
- (ii') Furthermore, if there exists optimal arguments I^* and f^* , then
for any $y \in \mathbb{V}_b(t, x)$, $\exists$ optimal α^* : $y = Y_t^{t, x, \alpha^*}$ and $Y_s^{t, x, \alpha^*} \in \mathbb{V}_b(s, X_s^*)$

Moving Scalarization: Given λ , choose y^λ solving $\sup_y \lambda \cdot y$. By verification, construct α^* , X^* , Y^* , and since $Y_t^* \in \mathbb{V}_b(t, X_t^*)$, α^* is optimal for $n(t, X_t^*, Y_t^*)$.

Mean-Variance: Introduce $\Psi(y_1, y_2) = (y_1, y_2 - y_1^2)$ and $\tilde{\mathbb{V}}(t, x) = \{\Psi(y) : \forall y \in \mathbb{V}(t, x)\}$

Then, $\sup_{y \in \mathbb{V}(t, x)} \Psi(y) = \sup_{\tilde{y} \in \tilde{\mathbb{V}}(t, x)} \tilde{y}^1 - \frac{\lambda}{2} \tilde{y}^2$ which is linear! Then, dynamic mean-variance

$V_t \doteq \text{ess}_\alpha \sup \left\{ \mathbb{E}[X_T^\alpha | \mathcal{F}_t] - \frac{\Lambda(t, X_{[0, t]}^*)}{2} \text{Var}(X_T^\alpha | \mathcal{F}_t) \right\}$ is time-consistent.

$$\alpha^* = -X_t^* + x_0 + \frac{1}{\lambda} e^T, \quad \Lambda(t, x_{[0, t]}) = \frac{\lambda e^{T-t}}{e^T - \lambda(x_t - x_0)}$$

$$V_t = \frac{1}{2} (1 + e^{-(T-t)}) X_t^* + \frac{1}{2} (1 - e^{-(T-t)}) x_0 + \frac{e^T}{2\lambda} (1 - e^{-(T-t)})$$

Set Values & Set Hamiltonian of Games

$$V_s^{t,x,\vec{a},i} = g^i(X_T^{t,x}) + \int_s^T h^i(r, X_r^{t,x}, Z_r^{t,x,\vec{a},i}, \vec{a}_r) dr - \int_s^T Z_r^{t,x,\vec{a},i} dB_r$$

$$h^i(t,x,z,\vec{a}) := f^i(t,x,\vec{a}) + z b(t,x,\vec{a})$$

Global

$$J^i(t,x,\vec{a}) \geq J^i(t,x,(\vec{a}^{-i}, \alpha^i))$$

$$V(t,x) := \{ \tilde{J}(t,x,\vec{a}^*) : \forall \alpha^* \in E(t,x) \}$$

Local

$$h^i(t,x,z,\vec{a}) \geq h^i(t,x,z,(\vec{a}^{-i}, a^i))$$

$$H(t,x,z) := \{ \tilde{h}(t,x,z,\vec{a}^*) : \forall \vec{a}^* \in E(t,x,z) \}$$

• Assumption: $H(t,x,z)$ non-empty & continuous

Seperability

Defn H is L -seperable : $H(t, x, z) = cl\{H^n(t, x, z) : \forall n\}$ where H^n measurable in (t, x) , L -Lipschitz in z .

Theorem

$$V(t, x) = cl\{Y_{t,x}^I : I \in \mathcal{I}_t\}$$

$$Y_s^{t,x,I,i} = g^i(X_T^{t,x}) + \int_s^T H^{I,i}(r, X_r^{t,x}, Z_r^{t,x,I,i}) dr - \int_s^T Z_r^{t,x,I,i} dB_r$$

$$H^I(t, x, z) \doteq H^{I(t,x,\cdot)}(t, x, z)$$

Corollary

Game can be embedded into a control problem,
where actions select which equilibrium to follow.

$$f(t, x, z, \alpha) \doteq H^\alpha(t, x, z)$$

$$\text{PDE} \sup_{n^T \xi = 0} \sup_{\bar{a} \in \mathcal{E}(\cdot, \partial_x V + \xi)} n^T \left[\partial_t V(\cdot) + h(\cdot, \partial_x V, a) + \frac{1}{2} \text{tr}(\partial_{xx} V) - \mathcal{K}^\xi(\cdot) \right](t, x, y) = 0$$

Theorem

(i) Suppose $V \in C^{1,2}$. Then V is a classical solution to **PDE**

(ii) Suppose $U \in C^{1,2}$ and is a classical solution to **PDE**. Then $U = V$.

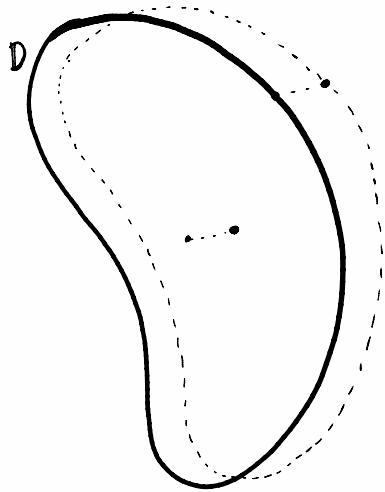
(ii') Furthermore, if there exists $I^*(t, x, y) \in \mathcal{E}(\cdot, \partial_x V + \xi^*)$ an optimal argument,

then $\alpha_t^* = I^*(t, X_t, \Upsilon^*)$ is an equilibrium and $Y_t^{**} = \Upsilon_t^* \in \mathbb{V}_b(t, X_t)$.

Thank

You

Example 1



$$V(x) = a(x) + D$$

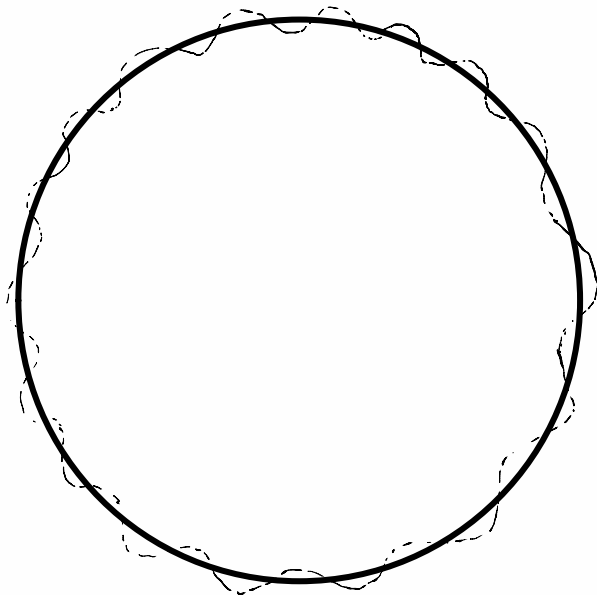
$$\partial_x V(x, y) = n n^T \nabla_x a(x)$$

Example 2

$$V(x) = B(a(x), R(x))$$

$$\partial_x V = n n^T \nabla_x a + n \nabla_x R$$

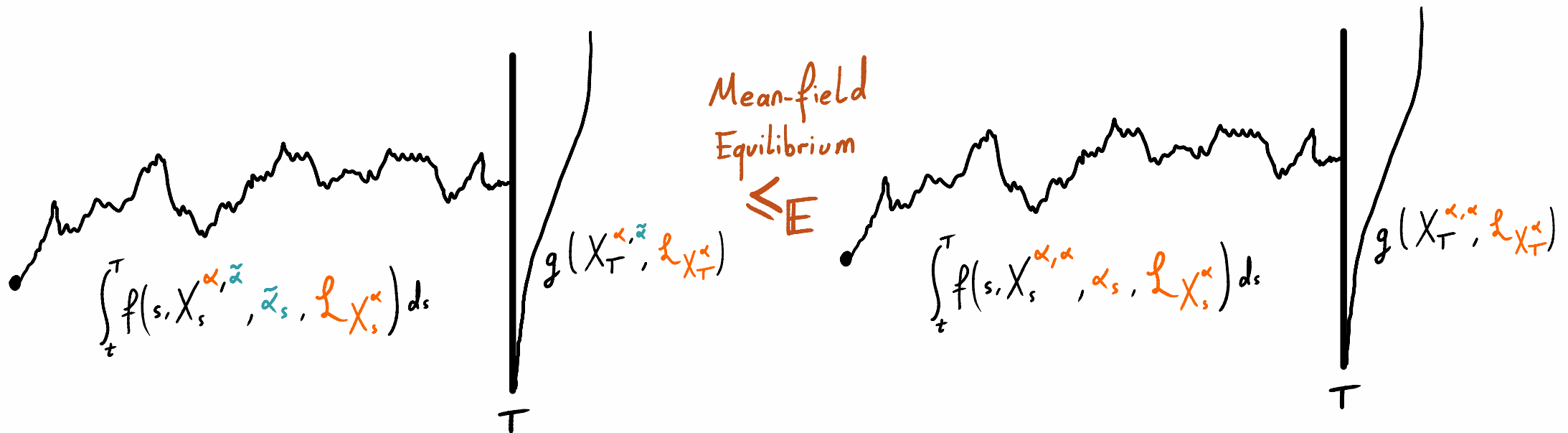
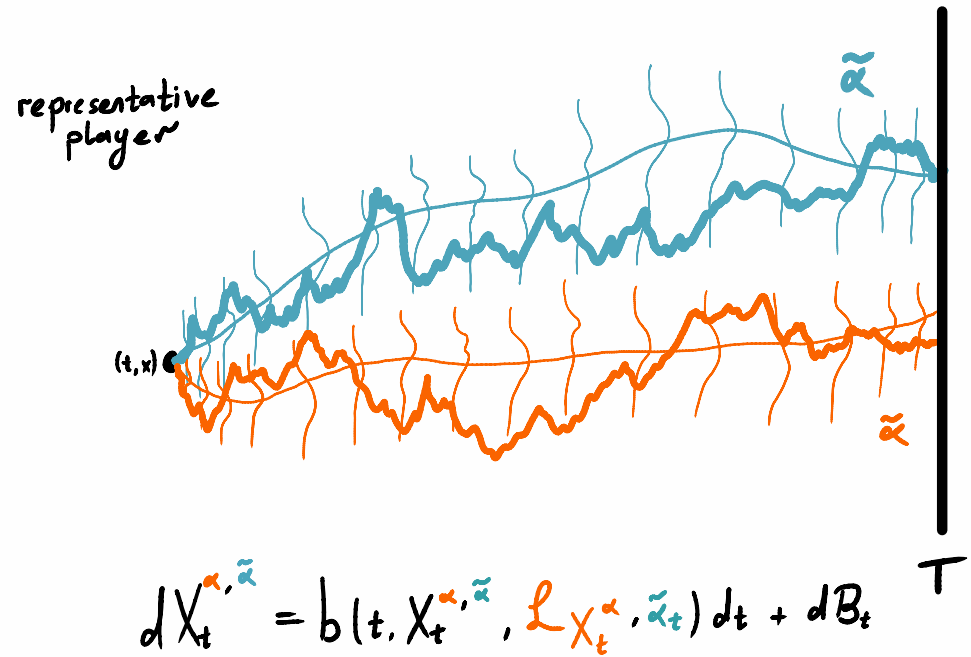
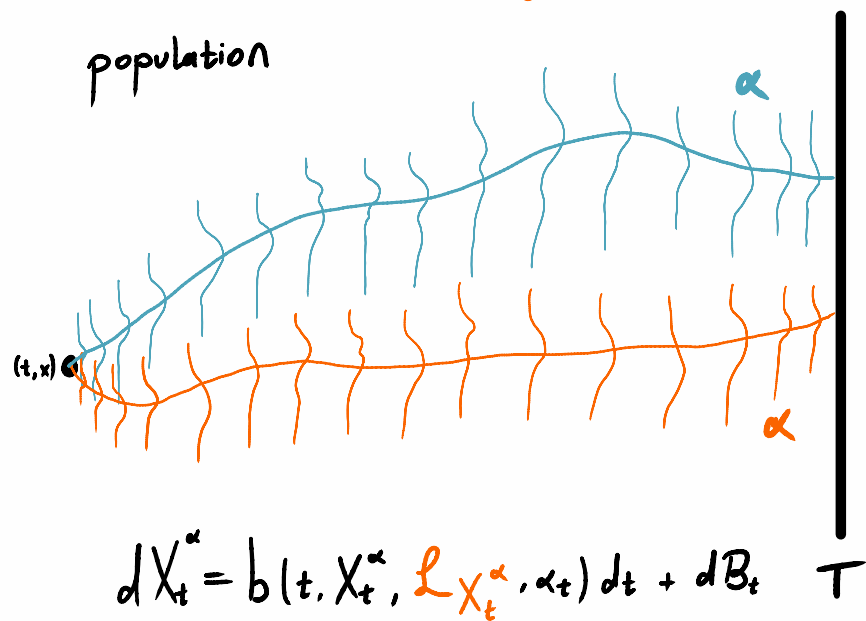
Example 3



$$V_b(x) = \left\{ [1 + x \cos(1/x + m\theta)] (\cos \theta, \sin \theta) : \forall \theta \right\}$$

- continuous
- NOT Differentiable
- n_v is not continuous

Mean-field Games



Mean-field Games

$$V(t, \mu) = \bigcap_{\epsilon > 0} \{ \text{values for all } \epsilon\text{-mean-field equilibria} \} \subset \mathbb{R}^d \rightarrow \mathbb{R}$$

- Convergence of N-player set value to mean-field

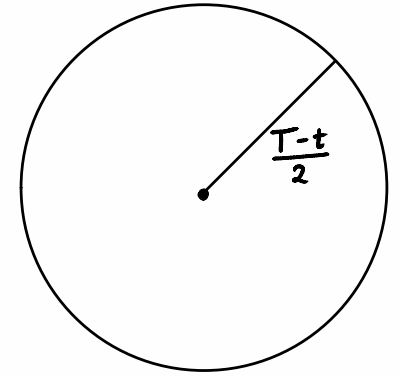
$$\lim_{N \rightarrow \infty} V^N(t, \mu^N) = V(t, \mu) \quad \text{when } \mu^N \rightarrow \mu$$

- Dynamic Programming Principle

$$V(t, \mu) = \{ \text{all mean-field equilibria } \alpha^* \text{ w/ terminal } (T_0, \phi) \text{ where } \phi \in V(T_0, \mu_{T_0}^{\alpha^*}) \}$$

(?) Itô's Formula $\Rightarrow$ Master Equation without uniqueness

Geometric Properties

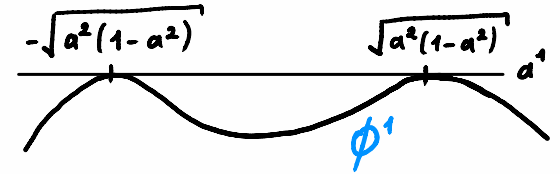


Theorem $\mathbb{V}(t, x)$ is a compact, convex set.

Simple Ball: $N=2, A=[0,1], b=0, g=0, f(a^1, a^2) = (\phi^1(a^1, a^2) + a^2, a^1)$

$$H(t, x, z) = H = \{ h(a^*) : \forall a^* \in E \} = B_b(1/2, 1/2)$$

$$\mathbb{V}(t) = \text{cl} \left\{ E \left[\int_t^T H^I ds \right] : \forall I \in \mathcal{I}_t \right\}$$



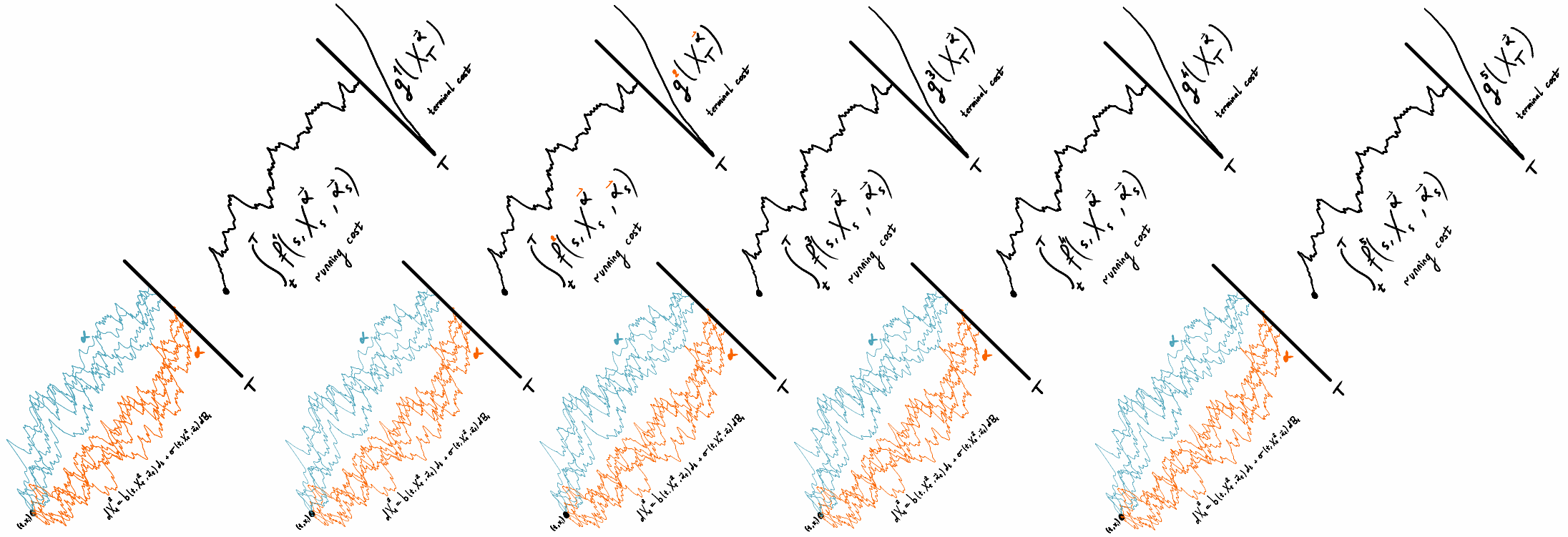
Deterministic: $b(a^1, a^2) = (a^1(2a^2-1), 0) = f(a^1, a^2), g(x^1, x^2) = (0, |x^1|^2)$

$$X_s^{t,x,\alpha} = x + \int_t^s b(\alpha_r) dr, \quad \mathcal{T}(t, x, \alpha) = g(X_T^{t,x,\alpha}) + \int_t^T f(\alpha_r) dr$$

not convex!

$$\mathcal{T}^1(t, x, \alpha) = \int_t^T \alpha_s^1(2\alpha_s^2-1) ds, \quad \mathcal{T}^2(t, x, \alpha) = (X_T^{t,x,\alpha})^2 = (x^1 + \mathcal{T}^1(t, x, \alpha))^2$$

Stochastic Games



Nash Equilibrium (Stable points)

$$\mathbb{E} \left[g^i(X_T^{\vec{a}^{-i}, \vec{a}^i}) + \int_t^T f^i(s, X_s^{\vec{a}^{-i}, \vec{a}^i}, \vec{a}_s^i) ds \right] \leq \mathbb{E} \left[g^i(X_T^{\vec{a}}) + \int_t^T f^i(s, X_s^{\vec{a}}, \vec{a}_s) ds \right]$$

Future Directions

- Master Equation for Mean-field Games
- Viscosity Solutions
- Surfaces with higher co-dimensions
- Numerical Methods (BSDE representation for set-PDEs & Forward dual representation)
- Applications to Portfolio Optimization with Multiple Objectives
- Continuous-time Reinforcement Learning
- Foundations for Player Design in Games