Set Values of Mean Field Games

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Outline

Set Valued Frameworks

- $\mathbb{V}$: parameters $(t, x, \mu, \cdots) \to$ collections
  - Geometric Surface Evolutions $\left[\partial_t \mathbb{V}(t) = h(t, y, \mathbf{n}, \partial_y \mathbf{n})\right]$
    - Mean curvature flows, crystal formations, image processing
  - Stochastic Viability & Target Problems
  - Dynamic Risk Measures
  - N-player Games
  - Multivariate Control Problems
  - Mean-field Games

Multivariate Control Problem

$$\mathbb{V}(t,x) := \Big\{ J(t,x,\alpha) : \forall \alpha \Big\} \in \mathbb{R}^m$$

- Dynamic Programming Principle i.e. Time-consistency
- Itô formula
- PDE (Hamilton-Jacobi-Bellman)

Mean-field Games

$$\mathbb{V}(t,\mu) := \Big\{ J(t,\mu,\alpha) : \forall \alpha \text{ equilibrium} \Big\}$$

- Dynamic Programming Principle
- Convergence of $\mathbb{V}^N$ to $\mathbb{V}$
- PDE (?) (Master Equation)

Primitive Structure

N-Player Game

– Actions

α^i: controls of the player

– Dynamics

X_t^i: state of individual player

$$\mu_t^N = \frac{1}{N}\sum_j \delta_{X_t^i}$$

– Cost

$J_i(\vec{\alpha})$: cost of the player

– Equilibrium

Nash Equilibrium:

$$J_i(\vec{\alpha}^*) \leq J_i(\vec{\alpha}^*, \alpha^i)$$

Mean Field Game

– Actions

α: control of the population

$\tilde{\alpha}$: control of a player

– Dynamics

X_t^α: state of the population

$$\mu_t^\alpha = \mathcal{L}_{X_t^\alpha}$$

$X_t^{\alpha,\tilde{\alpha}}$: state of a player

– Cost

$J(\alpha, \tilde{\alpha})$: cost of a player

– Equilibrium

Mean field Equilibrium:

$$J(\alpha^*, \alpha^*) \leq J(\alpha^*, \tilde{\alpha})$$

Examples to Equilibria without Dynamics

N-Player Game

$A = \{0, 1\}$, two player.

(J_1, J_2)	$a_1 = 0$	$a_1 = 1$
$a_2 = 0$	(2,2)	(3,1)
$a_2 = 1$	(1,3)	(1,1)

Mean Field Game

$A = [0, 1]$,

$$J(a, \tilde{a}) = 1 - a\tilde{a}$$

$a = 0$ and $a = 1$ are equilibria.

Examples of Different Structures

- Games on sparse & dense graphs
- Games of timing (optimal stopping)
- Cooperative Equilibriums
- Games with a major player
- Games with clusters
- Ergodic games
- Finite player games

$\vdots$

- Finite/continuous state/time
- Common noise
- Diffusion with jumps
- Information available to players
- State/path dependent dynamics
- State/path dependent costs
- Non-symmetric costs for players

$\vdots$

Classical References

Huang-Malhamé-Caines (2006)
Lasry-Lions (2007)
Lions (2008)
Cardaliaguet (2010)
Carmona-Delarue (2018)
⋮

References on Master Equation Approach **(under monotonicity condition)**

Buckdahn-Li-Peng-Rainer (2017)
Chassagneux-Crisan-Delarue (2014)
Cardaliaguet-Delarue-Lasry-Lions (2019)
Gangbo-Mészáros-Mou-Zhang (2021)
⋮

Objective & Related Literature

Our objective is to study dynamic set value $\mathbb{V}(t, \mu)$:

(i) Establish time-consistency (DPP)

(ii) characterize $\mathbb{V}$ as an appropriate limit of $\mathbb{V}^N$ (Convergence)

- Set Value is by definition unique and exists.
- We consider convergence of Set Values instead of individual equilibriums.

Settings

- Discrete time & space | state dependent controls | homogeneous controls
- Discrete time & space | path dependent controls | heterogeneous controls
- Continuous time & space | state dependent controls | homogeneous controls

Related to DPP

Feinstein-Rudloff-Zhang (2020)

Related to Convergence

Cardaliaguet-Delarue-Lasry-Lions (2019)
Lacker (2014, 2020)
Lacker-Flem (2021)
Possamaï-Tangpi (2021)
Djete (2021)

⋮

Discrete State Dependent Setup

- $\mathbb{S}$, finite state space.
- $\mathbb{T} = \{0, \cdots, T\}$, finite time steps.
- $\mathbb{A}$ is the set of possible values of the control.
- Transition function:

$$q(t, x, \nu, a; \tilde{x}) : \mathbb{T} \times \mathbb{S} \times \mathcal{P}(\mathbb{S}) \times \mathbb{A} \times \mathbb{S} \to (0, 1]$$

$$\text{where} \quad \sum_{\tilde{x} \in \mathbb{S}} q(t, x, \nu, a; \tilde{x}) = 1$$

Given $(t, \mu, \alpha; \tilde{\alpha}, x)$,

$$\text{(Population)} \quad \mu^{\alpha}_{s+1}(\tilde{x}) = \sum_{x \in \mathbb{S}} \mu^{\alpha}_s(x) q(s, x, \mu^{\alpha}_s, \alpha(s, x, \mu^{\alpha}_s); \tilde{x})$$

$$\text{(Player)} \quad \mathbb{P}^{\mu^{\alpha}; t, x, \tilde{\alpha}}(X_{s+1} = x | X_s) = q(s, X_s, \mu^{\alpha}_s, \tilde{\alpha}(s, X_s, \mu^{\alpha}_s); x)$$

$$J(t, \mu, \alpha; \tilde{\alpha}, x) \doteq \mathbb{E}^{\mathbb{P}^{\mu^{\alpha}; t, x, \tilde{\alpha}}} \left[G(X_T, \mu^{\alpha}_T) + \sum_{t}^{T-1} F(\cdots) \right]$$

Definition

We say $\alpha^* \in \mathcal{M}(t, \mu)$ a MFG Equilibrium if

$$J(t, \mu, \alpha^*; \alpha^*, x) \leq J(t, \mu, \alpha^*; \tilde{\alpha}, x), \quad \forall x, \tilde{\alpha}$$

Definition

$$\mathbb{V}(t, \mu) = \left\{ J(t, \mu, \alpha^*; \alpha^*, \cdot) : \text{ for all } \alpha^* \in \mathcal{M}(t, \mu) \right\}$$

Given $(t, \vec{x}, \vec{\alpha})$,

$$\mathbb{P}^{t,\vec{x},\vec{\alpha}}(\vec{X}_{s+1} = \vec{x}' | \vec{X}_s) = \prod_j q(s, X^j_s, \mu^N_{\vec{X}_s}, \alpha^j(X^j_s, \mu^N_{\vec{X}_s}); x'_j)$$

$\vec{X} = (X^1, \cdots, X^N)$ is the cannonical process on $(\mathbb{T} \times \mathbb{S})^N$.

$$J^N_i(t, \vec{x}, \vec{\alpha}) \doteq \mathbb{E}^{\mathbb{P}^{t,\vec{x},\vec{\alpha}}} \left[G(X^i_T, \mu^N_{\vec{X}_T}) + \sum_t^{T-1} F(\cdots) \right]$$

Definition

We say $\vec{\alpha}^* \in \mathcal{M}^N(t, \vec{x})$ if

$$J^N_i(t, \vec{x}, \vec{\alpha}^*) \leq J^N_i(t, \vec{x}, \vec{\alpha}^{*,-i}\tilde{\alpha}^i), \quad \forall i, \tilde{\alpha}$$

Definition

$$\mathbb{V}^N(t, \mu^N_{\vec{x}}) \doteq \left\{ (J^N_i(t, \vec{x}, \vec{\alpha}^*))_i : \text{ for all } \vec{\alpha}^* \in \mathcal{M}^N(t, \vec{x}) \right\}$$

ϵ-MFG Equilibriums

ϵ-MFG Equilibrium: $J(t,\mu,\alpha^*;\alpha^*,x) - \epsilon \leq J(t,\mu,\alpha^*;\tilde{\alpha},x)$.

Define the Set Value as;

$$\mathbb{V}(t,\mu) \doteq \bigcap_{\epsilon>0} \mathbb{V}_\epsilon(t,\mu)$$

• This definition is analoguous to the standard control theory, as the value defined to be the infimum over ϵ-optimal controls:

$$v(t,x) = \lim_{\epsilon\to 0} J_\epsilon(t,x), \quad J_\epsilon(t,x) = \inf_{\alpha^\epsilon} J(t,x,\alpha^\epsilon)$$

• It is possible that there is no optimal control.

Stability Issue

N-Player Game

$\mathbb{A} = \{0, 1\}$, two player.

(J_1, J_2)	$a_1 = 0$	$a_1 = 1$
$a_2 = 0$	(2,2)	(3,1-δ)
$a_2 = 1$	(1-δ,3)	(1,1)

Mean Field Game

$\mathbb{A} = [0, 1]$,

$$J(a, \tilde{a}) = 1 - (a + \delta)\tilde{a}$$

Only $a = 1$ is an equilibrium.

- Extension to ϵ-equilibria is crucial to have stability hence convergence.

Information that control depends on

- $\alpha = \alpha(t, X_t^i, \mu_t^N)$ State Dependent
  [DPP holds / Convergence holds]
- $\alpha = \alpha(t, X_{[0,t]}^i, \mu_{[0,t]}^N)$ Path Dependent
  [DPP holds / Convergence holds (Continuous?)]
- $\alpha = \alpha(t, X_1, \cdots, X_N)$ Full information ($\rightarrow$ weak MFG)
- $\cdots$

For games, difference is crucial because when equilibrium is not unique, set values depend on the choice.

- $\mathbb{V}_{state} \neq \mathbb{V}_{path}$ •

Notes on Convergence

• In MFG, there is only one control that population use.
However, we need 'same' type of controls for both NPG and MFG:

- $\mathbb{V}(t,\mu)$ is characterized as the limit fo homogeneous controls $(\alpha^1 = \cdots = \alpha^N)$
- Without this restriction, limit of NPG is characterized by using relax control for MFG. $\left(\int_{\mathbb{A}} q(s,x,\mu^\gamma, a; \tilde{x})\gamma(s,x,da)\right)$

• Introduce corresponding Set Values;

$$\mathbb{V}_\epsilon^{N,hom}(t,\mu^N_{\vec{x}}) \quad \text{and} \quad \mathbb{V}_\epsilon^{relax}(t,\mu)$$

An Important Observation

Introduce

$$\Lambda^N(t,\vec{x}_\cdot,\vec{\alpha}) = \frac{1}{N}\sum_{j=1}^{N}\delta_{(x^j_\cdot,\alpha^j)} \in \mathcal{P}(\mathbb{X}_t \times \mathcal{A}_{path})$$

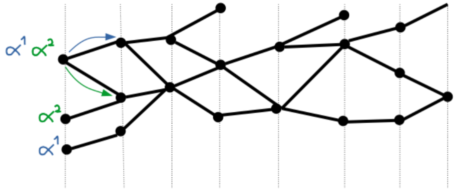
There exists J^N such that

$$J^N_i(t,\vec{x}_\cdot,\vec{\alpha}) = J^N(t,\Lambda^N(t,\vec{x}_\cdot,\vec{\alpha}); x^i_\cdot,\alpha^i)$$

Next, we introduce MFG corresponding to $\Lambda \in \mathcal{P}(\mathbb{X}_t \times \mathcal{A}_{path})$ with marginal and conditional distribution;

$$\mu^\Lambda(x_\cdot) \doteq \int_{\mathcal{A}_{path}} \Lambda(x_\cdot, d\alpha), \qquad \Lambda_{x_\cdot}(\alpha) \doteq \frac{\Lambda(x_\cdot,\alpha)}{\mu^\Lambda(x_\cdot)}$$

Global Formulation for MFG



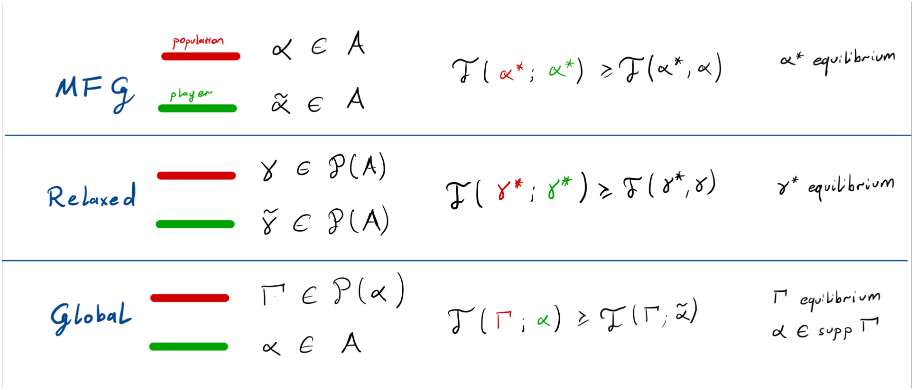
MFG: Given $(t, \Lambda; \tilde{\alpha}, x_\cdot)$, introduce a mapping $\Lambda_s : \mathbb{T}_t \to \mathcal{P}(\mathbb{X} \times \mathcal{A}_{path})$,

$$\text{(Population)} \qquad \Lambda_{s+1}(x_\cdot, \alpha) \doteq q(s, x_s, \mu_s^\Lambda, \alpha(s, x_\cdot, \mu^\Lambda); x_{s+1})\Lambda_s(x_\cdot, \alpha)$$

$$\text{(Player)} \qquad \mathbb{P}^{\mu^\Lambda; t, x_\cdot, \tilde{\alpha}}(X_{s+1} = x | X_\cdot) = q(s, X_s, \mu_s^\Lambda, \tilde{\alpha}(s, X_\cdot, \mu_s^\Lambda); x)$$

$$J(t, \Lambda; \tilde{\alpha}, x_\cdot) = \mathbb{E}^{\mathbb{P}^{\mu^\Lambda; t, x_\cdot, \tilde{\alpha}}} \left[G(X_T, \mu_T^\Lambda) + \sum_{s=t}^{T-1} F(\cdots) \right]$$

Global Formulation and Relax Controls



- NPG is naturally connected with $\Lambda^N = (1/N)\sum_j \delta_{(x^j, \alpha^j)}$.

Theorem

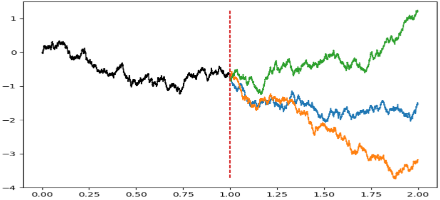
$$\mathbb{V}^{global}(t, \mu) = \mathbb{V}^{relax}(t, \mu)$$

Time Consistency - Dynamic Programming Principle

- Define cost function up to time $T_0 \leq T$ with a given terminal cost ψ.
- Introduce appropriate MFG Equilibrium $\mathcal{M}(T_0, \psi, t, \mu)$.

Theorem

$$\mathbb{V}(t,\mu) = \Big\{ J(T_0, \psi; t, \mu, \alpha^*; \alpha^*, \cdot) : \text{ for some } \alpha^* \in \mathcal{M}(T_0, \psi, t, \mu) \text{ and } \psi \in \mathbb{V}(T_0, \mu^{\alpha^*}_{T_0}) \Big\}$$



- Holds for $\mathbb{V}^{raw}$, $\mathbb{V}$, $\mathbb{V}^{relax}$.
- DPP is central for PDE approach, which is an ongoing project.

Convergence Results

Theorem

For $\mu^N_{\vec{x}} \Rightarrow \mu$,

$$\bigcap_{\epsilon>0} \liminf_{N\to\infty} \mathbb{V}^{N,hom}_{\epsilon}(t,\mu^N_{\vec{x}}) = \mathbb{V}(t,\mu) = \bigcap_{\epsilon>0} \limsup_{N\to\infty} \mathbb{V}^{N,hom}_{\epsilon}(t,\mu^N_{\vec{x}})$$

Theorem

For $\mu^N_{\vec{x}} \Rightarrow \mu$,

$$\bigcap_{\epsilon>0} \liminf_{N\to\infty} \mathbb{V}^{N}_{\epsilon}(t,\mu^N_{\vec{x}}) = \mathbb{V}^{relax}(t,\mu) = \bigcap_{\epsilon>0} \limsup_{N\to\infty} \mathbb{V}^{N}_{\epsilon}(t,\mu^N_{\vec{x}})$$

Setup

N-player game

$$X_s^i = x_i + \int_t^s b(r, X_r^i, \mu_r^N, \alpha_i(r, X_r^i, \mu_r^N))ds + B_s^i - B_t^i, \quad \mu_r^N = \frac{1}{N}\sum_j \delta_{X_r^j}$$

$$J_i(t, \vec{x}, (\alpha \tilde{\alpha}^i)) = J(t, \mu_t^N, \alpha; x_i, \tilde{\alpha}) = \mathbb{E}\Big[G(X_T^i, \mu_T^N) + \int_t^T F(\cdots)\Big]$$

$$\alpha^* \in \mathcal{M}^\epsilon(t, \mu^N) : \int_{\mathbb{R}^d} [J(t, \mu^N, \alpha^*; x, \alpha^*) - \inf_{\tilde{\alpha}} J(t, \mu^N, \alpha^*; x, \tilde{\alpha})]\mu^N(dx) \leq \epsilon$$

Mean Field game

$$X_s^\alpha = \xi + \int_t^s b(r, X_r^\alpha, \mu_r, \alpha(r, X_r^\alpha, \mu_r))ds + B_s - B_t, \quad \mu_r^\alpha = \mathcal{L}_{X_s^\alpha}$$

$$X_s^{\alpha,\tilde{\alpha}} = x + \int_t^s b(r, X_r^{\alpha,\tilde{\alpha}}, \mu_r^\alpha, \tilde{\alpha}(r, X_r^{\alpha,\tilde{\alpha}}, \mu_r^\alpha))ds + B_s - B_t$$

$$J(t, \mu, \alpha; x, \tilde{\alpha}) = \mathbb{E}\Big[G(X_T^{\alpha,\tilde{\alpha}}, \mu_T^\alpha) + \int_t^T F(\cdots)\Big]$$

$$\alpha^* \in \mathcal{M}^\epsilon(t, \mu) : \int_{\mathbb{R}^d} [J(t, \mu^N, \alpha^*; x, \alpha^*) - \inf_{\tilde{\alpha}} J(t, \mu, \alpha^*; x, \tilde{\alpha})]\mu(dx) \leq \epsilon$$

Main Results

We define the Set Value as

$$\mathbb{V}_\epsilon(t,\mu) \doteq \left\{\varphi \in C_{Lip}(\mathbb{R}^d) : \exists \alpha^* \in \mathcal{M}^\epsilon(t,\mu) \text{ s.t. } \int_{\mathbb{R}^d} |\varphi(x) - J(t,\mu,\alpha^*;x,\alpha^*)|\mu(dx) \leq \epsilon\right\}$$

Theorem

$$\bigcap_{\epsilon>0} \liminf_{N\to\infty} \mathbb{V}_\epsilon^{N,hom}(t,\mu_{\vec{x}}^N) = \mathbb{V}(t,\mu) = \bigcap_{\epsilon>0} \limsup_{N\to\infty} \mathbb{V}_\epsilon^{N,hom}(t,\mu_{\vec{x}}^N)$$

Theorem

$$\mathbb{V}(t,\mu) = \bigcap_{\epsilon>0} \Big\{\varphi \in C_{Lip}(\mathbb{R}^d) : \exists(\psi,\alpha^\epsilon) \; s.t. \; \alpha^\epsilon \in \mathcal{M}^\epsilon(T_0,\psi,t,\mu) \text{ and } \psi \in \mathbb{V}_\epsilon(T_0,\mu_{T_0}^{\alpha^\epsilon}) \; \int_{\mathbb{R}^d} |\varphi(x) - J(T_0,\psi;t,\mu,\alpha^\epsilon;x,\alpha^\epsilon)|\mu(dx) \leq \epsilon\Big\}$$

References

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