FORTUIN-KASTELEYN PERCULATION MODEL

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Abstract. This is a presentation on Fortuin-Kasteleyn Percolation Model, focused on proving positive association and monotonicity properties. Entire presentation follows [1], and not self-explanatory.

1. Brief Introduction to General Construction

We are interested in finite subgraph $G = (V, E)$ of $\mathbb{Z}^d$ where edges are formed by nearest neighbourhoods. Moreover, attribute a *spin* variable from Σ to each vertex. Hamiltonian is given by;

$$\mathcal{H}^f_G(\sigma) = -\sum_{xy\in E} \sigma_x \cdot \sigma_y \tag{1.1}$$

Identify each σ_x by a reference measure σ_0 on Σ, and endow Σ^V with product measure. Gibbs measure is then given by[1]

$$\mu^f_{G,\beta}[f] = \frac{\int_{\Sigma^V} f(\sigma) \exp[-\beta\mathcal{H}^f_G(\sigma)]d\sigma}{\int_{\Sigma^V} \exp[-\beta\mathcal{H}^f_G(\sigma)]d\sigma} \tag{1.2}$$

2. Potts Model

Let Σ has q elements with the property that

$$a\cdot b = \begin{cases} 1 & \text{if} \quad a = b \\ -\frac{1}{q-1} & \text{otherwise} \end{cases} \tag{2.1}$$

and endow with uniform measure. Therefore, since hamiltonian is ferromagnetic, spins are favored by probability $1/q$ while expected is constant 0.

3. Fortuin-Kasteleyn Percolation Model

Aim of the Percolation Model is to embed spin interactions to a graph, where distribution is carried to a binary model of open/close edges. Surely distribution of edges cannot be independent, as it was independent for the vertices of Potts Model. However, measure can be simplified by determining a *weight* q for a number of clusters.

$$\phi^\xi_{G,p,q}[w] = \frac{p^{o(w)}(1-p)^{c(w)}q^{k(w^\xi)}}{Z^\xi_{G,p,q}} \tag{3.1}$$

If $q = 1$, it is called Bernoulli percolation. Let me state a calculation

$$\phi^\xi_{G,p,q}[w = 1|w_{E-\{e\}} = \psi] = \begin{cases} p & \text{if } x \leftrightarrow y \text{ in } \psi^\xi \\ \frac{p}{p+q(1-p)} & \text{otherwise} \end{cases} \tag{3.2}$$

Note that if they are already in the same cluster then they are connected in an independent manner. Otherwise, connecting clusters is penaltized. Roughly, each cluster forms a subgraph behaving as a Bernoulli model.

Coupling between Fortuin-Kasteleyn Percolation Model and Potts Model is given by

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[1]Boundary conditions are given by $\mu^f_{G,\beta}[\cdot|\sigma_x = b, x \in \partial G]$

Proposition 1. *If w is distributed according to $\phi^0_{G,p,q}$, assign each cluster an i.i.d. spin from reference measure of Potts model. Then σ is distributed according to q-state Potts model $\mu^f_{G,\beta,q}$ where*

$$\beta = -\frac{q-1}{q}\log(1-p) \tag{3.3}$$

Sketch. (Not to present.) For each cluster, there exists q states equally likely. Hence, measure on product space of (w,σ) is given by independent part of percolation measure. Summing over compatible w's, role of β can be attributed to the edges between clusters(of fixed σ), where sum over clusters add up to 1 since they were independent.

Therefore, correlations can be respresented as

$$\mu^f_{G,\beta,q}[\sigma_x \cdot \sigma_y] = \phi^0_{G,p,q}[x \leftrightarrow y] \qquad \text{and} \qquad \mu^b_{G,\beta,q}[\sigma_x \cdot b] = \phi^1_{G,p,q}[x \leftrightarrow y] \tag{3.4}$$

4. Positive Association and Monotonicity

Since percolation model is binary, it yields a natural partial ordering for configurations. This allows us to define events that is closed under increasing configurations.

Definition 1. Event $\mathcal{A}$ is called *increasing* if $w \in \mathcal{A}$ and $w \leq w'$ implies $w' \in \mathcal{A}$.

Note 1. Smallest non-empty increasing event contains only the configuration where all edges are open, denoted as w^o. It is contained in every non-empty increasing event.

Definition 2. We say μ is *stochasitcally dominated* by ν if for any increasing event $\mathcal{A}$, $\mu[\mathcal{A}] \leq \nu[\mathcal{A}]$.

Consider the product space $\{0,1\}^E \times \{0,1\}^E$ with a probability measure P on $(w,\tilde{w})$.[2] Suppose the law of w is μ and the law of $\tilde{w}$ is ν. If $P[w \leq \tilde{w}] = 1$, then μ is stochastically dominated by ν:

$$\mu[\mathcal{A}] = P[w \in \mathcal{A}] = P[w \in \mathcal{A}, w \leq \tilde{w}] \leq P[\tilde{w} \in \mathcal{A}] = \nu[\mathcal{A}] \tag{4.1}$$

Now, for two Bernoulli percolation model with $p \leq p'$, it is straightforward to construct P. Assign each edge e an uniform $[0,1]$ random variables U_e and define $w, \tilde{w}$ as follows

$$w_e = \begin{cases} 1 & \text{if } U_e \geq 1-p \\ 0 & \text{otherwise} \end{cases} \qquad \text{and} \qquad \tilde{w}_e = \begin{cases} 1 & \text{if } U_e \geq 1-p' \\ 0 & \text{otherwise} \end{cases} \tag{4.2}$$

In words, we consider a single space $\{0,1\}^E$ and a product measure given by U_e's. P is defined naturally in a way that it satisfies law of w and $\tilde{w}$, but more importantly, $w \leq \tilde{w}$ for any realization of U_e's.

In general, it is not straightforward to construct P. Next lemma provides a convenient criteria for existence of such measure P.

Lemma 1. *Suppose μ and ν are strictly positive measures on $\{0,1\}^E$ and for any increasing configurations $\psi \leq \psi'$,*

$$\mu[w_e = 1 | w_{|E-\{e\}} = \psi] \leq \nu[w_e = 1 | w_{|E-\{e\}} = \psi'] \tag{4.3}$$

Then there exists a measure P with $P[w \leq \tilde{w}] = 1$ such that w and w' have laws μ and ν.

Proof. Consider the space $\{0,1\}^E$ and associate each edge with an exponential clock[3] and a collection of independent uniform $[0,1]$ random variables $U_{e,k}$. Note that with probability 1, clocks ring at different times.

Define continuous time Markov chain $(w^t, \tilde{w}^t)$ as follows; for each ring of a clock, allow system jump to a new configuration, where jump probabilities only depending on the current state:

$$\begin{aligned} w^t_e &= \begin{cases} 1 & \text{if } U_{e,k} \geq \mu[w_e = 0 | w_{|E-\{e\}} = w^{t^-}_{|E-\{e\}}] \\ 0 & \text{otherwise} \end{cases} \\ \tilde{w}^t_e &= \begin{cases} 1 & \text{if } U_{e,k} \geq \nu[w_e = 0 | w_{|E-\{e\}} = w^{t^-}_{|E-\{e\}}] \\ 0 & \text{otherwise} \end{cases} \end{aligned} \tag{4.4}$$

[2] Note that it cannot be a product measure.

[3] A cumulative sequence of i.i.d. exponential variables, say $\left\{\sum^k \exp(1)\}\right\}_k$

Note that chains are irreducable, since all states are reachable. It is only left to show that μ and ν are stationary measures. Then law of w^t converges to μ and similarly $\tilde{w}^t$ converges to ν, moreover, they preserve their orderings hence we get $P[w \leq \tilde{w}] = 1$.

Let $p(\psi, \psi')$ be the jump probability.[4] It is 0, if ψ and ψ' differ by more than one edge. Hence, denote ψ_e as same as ψ expect the edge e. We need to show

$$\mu[\psi]p(\psi,\psi) + \sum_e \mu[\psi_e]p(\psi_e,\psi) = \mu[\psi] \tag{4.5}$$

Denote R_e the event clock e ringed first. Observe that $p(\psi_e, \psi|R_e) = p(\psi, \psi|R_e)$ because $\psi_{e|E-\{e\}} = \psi_{|E-\{e\}}$. This probability is then given by;

$$P(U_{e,k} \geq \mu[w_e = 1 - \psi_{|\{e\}}|w = \psi \text{ on } E - \{e\}]) = \frac{\mu[\psi]}{\mu[\psi] + \mu[\psi_e]} \tag{4.6}$$

Consider $p(\psi, \psi)$, i.e. chain stays at the same state.

$$p(\psi,\psi) = \sum_e P(R_e)\left(\frac{\mu[\psi]}{\mu[\psi] + \mu[\psi_e]}\right) \tag{4.7}$$

Hence 4.5 can be re-written as;

$$\sum_e P(R_e)\left(\mu[\psi]\frac{\mu[\psi]}{\mu[\psi] + \mu[\psi_e]} + \mu[\psi_e]\frac{\mu[\psi]}{\mu[\psi] + \mu[\psi_e]}\right) \tag{4.8}$$

Rest is a simple calculation by noting that $\sum_e P(R_e) = 1$ □

Theorem 1. (Positive Association)

- *For any increasing $\mathcal{A}$ and boundary conditions $\xi \leq \xi'$*
$$\phi^{\xi}_{G,p,q}[\mathcal{A}] \leq \phi^{\xi'}_{G,p,q}[\mathcal{A}] \tag{4.9}$$
- *For any increasing event $\mathcal{A}$ and any $p \leq p'$,*
$$\phi^{\xi}_{G,p,q}[\mathcal{A}] \leq \phi^{\xi}_{G,p',q}[\mathcal{A}] \tag{4.10}$$
- *(Fortuin-Kasteleyn-Ginibre Inequality) For any increasing events $\mathcal{A}$ and $\mathcal{B}$,*
$$\phi^{\xi}_{G,p,q}[\mathcal{A}]\phi^{\xi}_{G,p,q}[\mathcal{B}] \leq \phi^{\xi}_{G,p,q}[\mathcal{A} \cap \mathcal{B}] \tag{4.11}$$

Proof. By Lemma, it is sufficient to show that for $e = xy \in E$ and $\psi \leq \psi'$

$$\phi^{\xi}_{G,p,q}[w_e = 1|w|_{E-\{e\}} = \psi] \leq \phi^{\xi'}_{G,p,q}[w_e = 1|w|_{E-\{e\}} = \psi'] \tag{4.12}$$

which holds by 3.2 because if x and y are connected in ψ^{ξ}, they are connected in $(\psi')^{\xi}$ and furthermore in $(\psi')^{\xi'}$. If this is not the case, $p/(p + q(1-p)) \leq p$ hence inequality follows. For monotonicity,

$$\phi^{\xi}_{G,p,q}[w_e = 1|w|_{E-\{e\}} = \psi] \leq \phi^{\xi}_{G,p',q}[w_e = 1|w|_{E-\{e\}} = \psi'] \tag{4.13}$$

follows again by 3.2 immediately because

$$\frac{1}{1 + q\frac{1-p}{p}} \leq \frac{1}{1 + q\frac{1-p'}{p'}} \tag{4.14}$$

For FKG, define $\mu = \psi^{\xi}_{G,p,q}$ and $\nu = \mu[\cdot|\mathcal{B}]$ but ν is not strictly positive. We claim statement of lemma still holds. Markov chain restricted to $\mathcal{B}$ is still irreducable, and for the condition

$$\phi^{\xi}_{G,p,q}[w_e = 1|w|_{E-\{e\}} = \psi] \leq \phi^{\xi}_{G,p,q}[w_e = 1|w|_{E-\{e\}} = \psi'|\mathcal{B}] \tag{4.15}$$

we only need to check it for $\psi' \in \mathcal{B}$. It is sufficient because $w^o \in \mathcal{B}$(which surely satisfies $\psi \leq w^o$), and markov chain will jump only to states of $\mathcal{B}$: suppose markov chain is at the configuration $\psi' \in \mathcal{B}$, and suppose edge e ringed. If $\psi'_{|\{e\}}$ was closed, ψ'_e is also in $\mathcal{B}$. If it was open, and $\psi'_e \notin \mathcal{B}$ then the

[4]Jump probabilities are time-homogeneous, since clocks are memoryless.

probability it will be closed given $\mathcal{B}$ is 0 hence stays open surely. Finally, 4.15 holds since $\psi \leq \psi'$ preserves connectedness and $\psi^{\xi}_{G,p,q}[\mathcal{B}] \leq 1$. As a consequence,

$$\phi^{\xi}_{G,p,q}[\mathcal{A}] = \mu[\mathcal{A}] \leq \nu[\mathcal{A}] = \frac{\psi^{\xi}_{G,p,q}[\mathcal{A} \cup \mathcal{B}]}{\psi^{\xi}_{G,p,q}[\mathcal{B}]} \tag{4.16}$$

□

Corollary 1. *By comparison of boundary conditions, free and wired boundary conditions are extremal;*

$$\phi^{0}_{G,p,q}[\mathcal{A}] \leq \phi^{\xi}_{G,p,q}[\mathcal{A}] \leq \phi^{1}_{G,p,q}[\mathcal{A}] \tag{4.17}$$

for any increasing event $\mathcal{A}$.

Proof. Apply (4.9) with boundary conditions as singletons for free, and $\{\partial G\}$ for wired. □

Corollary 2. *The functions* $\beta \mapsto \mu^{f}_{G,\beta,q}[\sigma_x \cdot \sigma_y]$ *and* $\beta \mapsto \mu^{b}_{G,\beta,q}[\sigma_x \cdot b]$ *are non-decreasing.*

Proof. 3.4 and 4.10 implies the result, by noting that

$$\beta' = \frac{q-1}{q}\left(\frac{1}{1-p}\right) > 0 \tag{4.18}$$

□

References

[1] Hugo Duminil-Copin. *Lectures on the Ising and Potts models on the hypercubic lattice.* 2017. eprint: arXiv:1707.00520.