Set Values for Mean-Field Games & Set Valued PDEs

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- Set Valued $\mathbb{V}$: parameters $(t, x, \mu, \ldots)$ $\longrightarrow$  collections (instead of elements)

Set Valued Frameworks

- Geometric Surface Evolutions (phase transitions, crystal formations, image processing)
- Stochastic Viability & Target Problems
- Dynamic Risk Measures
- N-player Games
- Multivariate Control Problems
- Mean-field Games.

Multivariate Control Problem

$$\mathbb{V}(t,x) = \{ \mathcal{J}(t,x,\alpha) : \forall \alpha \}$$

- DPP

  i.e. time-consistency.

- PDE (HJB)

Mean-Field Games

$$\mathbb{V}(t,\mu) = \{ \mathcal{J}(t,\mu,\alpha) : \forall \alpha \text{ equilibrium} \}$$

- DPP

- Convergence $\left(\mathbb{V}^N \to \mathbb{V} \right)$

- PDE (Master Equation) ?

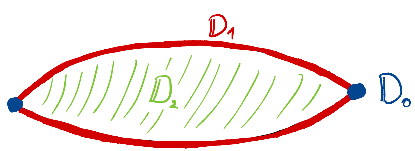
Set Valued Calculus

Defn $\mathbb{D} \in \mathcal{D}_2^m$ if it is closed and

- $\mathbb{D} = \bigcup_{1 \le k \le m} \mathbb{D}_k$, $\mathbb{D}_k$ is C^2-manifold with dimension k.
- $\exists\, n_1, \dots, n_{m-k} : \mathbb{D}_k \to \mathbb{R}^m$ basis for normal space & $\partial_y n(y)$ exists.

Defn $\mathbb{T}_{\mathbb{D}}(y) \doteq \{ \gamma'(0) \mid \gamma : \mathbb{R} \to \mathbb{D} ,\ \gamma(0) = y \}$ tangent space

$N_{\mathbb{D}}(y) \doteq \left(\mathbb{T}_{\mathbb{D}}(y) \right)^{\perp}$ normal space

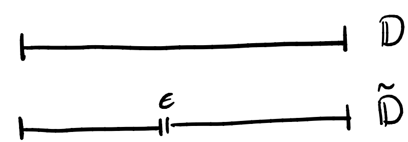


Given $\mathbb{V}(x) : \mathbb{R}^d \to \mathcal{D}_2^m$; $\mathcal{G}_k \doteq \{ (x,y) : x \in \mathbb{R}^d ,\ y \in \mathbb{V}_k(x) \}$ graph of $\mathbb{V}_k$

Continuity $\mathbb{V}(x) : \mathbb{R}^d \to \mathcal{D}_2^m$ is continuous under dl.

$$dl(\mathbb{D}, \tilde{\mathbb{D}}) \doteq \max_k d(\mathbb{D}_k, \tilde{\mathbb{D}}_k)$$

(Hausdorff) $d(\mathbb{D}, \tilde{\mathbb{D}}) = \inf \{ \epsilon > 0 : \tilde{\mathbb{D}} \subset \mathcal{O}_\epsilon(\mathbb{D}) \ \&\ \mathbb{D} \subset \mathcal{O}_\epsilon(\tilde{\mathbb{D}}) \}$.

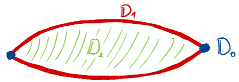


$f : \mathcal{G}_k \to \mathbb{R}$ is continuous

if $(x_n, y_n) \in \mathcal{G}_k \to (x,y) \in \mathcal{G}_k$ then $f(x_n, y_n) \to f(x,y)$

- $\mathbb{D} = \bigcup_{1\le k\le m} \mathbb{D}_k$

$\mathbb{D}_k$ is C^2-manifold with dimension k.



- $\mathcal{G}_k = \{(x,y) \mid x\in\mathbb{R}^d,\ y\in \mathbb{V}_k(x)\}$ graph of $\mathbb{V}_k$

$\mathbb{V}(x): \mathbb{R}^d \to \mathbb{D}_2^m$ is continuous under dl.

$$dl(\mathbb{D},\tilde{\mathbb{D}}) \doteq \max_k d(\mathbb{D}_k,\tilde{\mathbb{D}}_k)$$

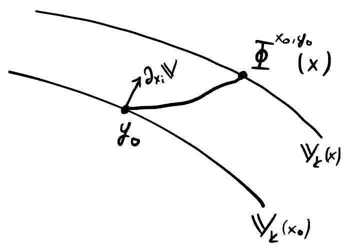
$f: \mathcal{G}_k\to\mathbb{R}$ is continuous

if $(x_n,y_n)\in\mathcal{G}_k \to (x,y)\in\mathcal{G}_k$ then $f(x_n,y_n)\to f(x,y)$

Differentiability

$\mathbb{V}(x): \mathbb{R}^d \to \mathbb{D}_2^m$ is differentiable if $\exists\ \Phi$ s.t.

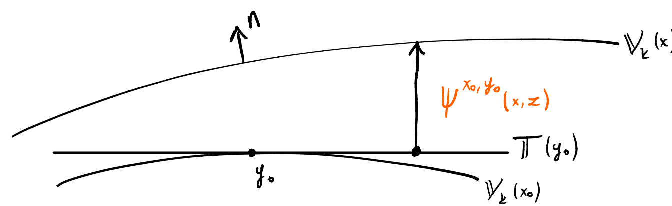
- $\Phi^{x_0,y_0}(x_0) = y_0$
- $\Phi^{x_0,y_0}(x) \in \mathbb{V}_k(x)$ if $y_0\in\mathbb{V}_k(x_0)$
- $\Phi^{x_0,y_0}(\cdot)$ is differentiable
- $\Phi_i'(x,y) \doteq \partial_{x_i}\Phi^{x,y}(x)$ is continuous in x, Lipschitz in y.



Moreover, $\exists$ continuous basis $n(x,y)$ for each k.

- $\partial_{x_i}\mathbb{V}(x,y) \doteq (N\cdot\Phi_i')(x,y)$ independent of Φ

Local Representation



$z\in\mathbb{R}^k$
$\psi\in\mathbb{R}^{m-k}$

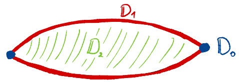
- $n \sim \partial_z\psi$ & $\partial_y n \sim \partial_{zz}\psi$

Lemma If $\mathbb{V}$ differentiable,

then ψ is differentiable in x & $\partial_z\psi$ is continous.

- $\mathbb{D} = \bigcup_{1 \le k \le m} \mathbb{D}_k$

$\mathbb{D}_k$ is C^2-manifold with dimension k.



- $\mathcal{G}_k = \{(x,y) \mid x \in \mathbb{R}^d, y \in \mathbb{V}_k(x)\}$ graph of $\mathbb{V}_k$

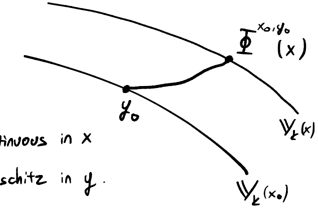
$\mathbb{V}(x) : \mathbb{R}^d \to \mathcal{D}_2^m$ is continuous under $d\!I$.

$$d\!I(\mathbb{D}, \tilde{\mathbb{D}}) \doteq \max_k d(\mathbb{D}_k, \tilde{\mathbb{D}}_k)$$

$f : \mathcal{G}_k \to \mathbb{R}$ is continuous

if $(x_n, y_n) \in \mathcal{G}_k \to (x,y) \in \mathcal{G}_k$ then $f(x_n, y_n) \to f(x,y)$

Differentiability



$\Phi_i'(x,y) \doteq \partial_{x_i} \Phi^{x,y}(x)$ is continuous in x

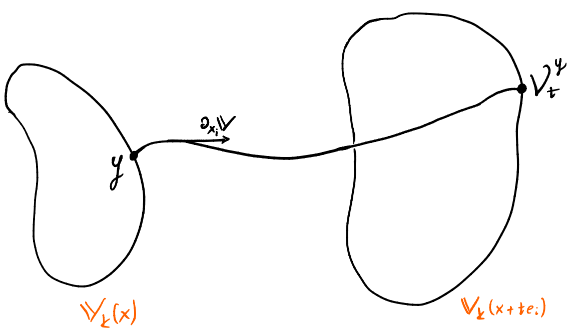
Lipschitz in y.

$$\partial_{x_i} \mathbb{V}(x,y) \doteq (N \cdot \Phi_i')(x,y) \quad \text{independent of } \Phi$$

Fundamental Theorem

$$V_t^i = y + \int_0^t \partial_{x_i} \mathbb{V}(x + s e_i, V_s^i)\, ds$$

$$\mathbb{V}_k(x + t e_i) = \{ V_t^y : y \in \mathbb{V}_k(x) \}$$



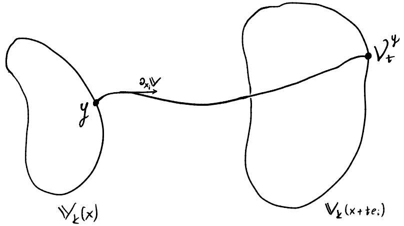
Defn $\partial_{x_i} f(x,y) \doteq \partial_s f(x + s e_i, V_s^i)|_{s=0}$

Ex $\partial_{x_j x_i} \mathbb{V}(x,y) \doteq \partial_{x_j}(\partial_{x_i} \mathbb{V}(x,y))$

Ex $\partial_x n$ plays a role in Itô's Formula.

Remark $\partial_{x_i x_j} \mathbb{V} \ne \partial_{x_j x_i} \mathbb{V}$

Fundamental Theorem



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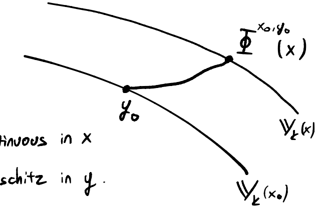
Differentiability

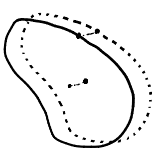


$\Phi_i'(x,y) \doteq \partial_{x_i}\Phi^{x,y}(x)$ is continuous in x, Lipschitz in y.

$\partial_{x_i}\mathbb{V}(x,y) \doteq (N \cdot \Phi_i')(x,y)$ independent of Φ

Example 1

$\mathbb{V}(x) = a(x) + \mathbb{D}$



$\Phi^{x_0,y_0}(x) = y_0 + a(x) - a(x_0)$

$\Rightarrow \partial_x\mathbb{V}(x,y) = N \cdot \partial_x a(x)$

Trivial Cases

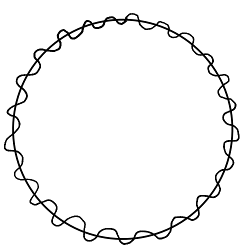
- $\mathbb{D}_0$ 
- Interior 

Example 2

$\mathbb{V}(x) = B(a(x), R(x))$. $\partial_x\mathbb{V} = nn^T\partial_x a + n\,\partial_x R$

Example 3

$\mathbb{V}(x) = \{[1 + x\cos(1/x + m\theta)](\cos\theta, \sin\theta) : \forall\theta\}$

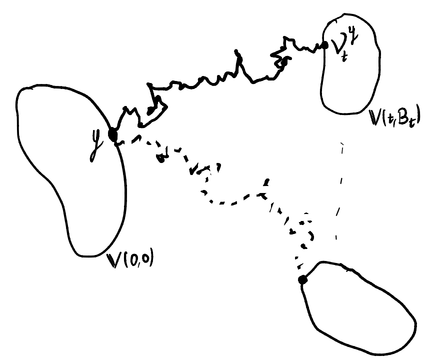


- continuous
- not differentiable
- n is not continuous.

Itô's Formula

$$V_t^y = y + \int_0^t \partial_t\mathbb{V} + \frac{1}{2}\partial_{xx}\mathbb{V} + K^\xi \, ds + \int_0^t \partial_x\mathbb{V} + \xi \, dB_s$$

$$\mathbb{V}_k(t, B_t) = \{V_t^y : \forall y \in \mathbb{V}_k(0,0)\}.$$



K^ξ

$\partial_x n \quad \partial_y n$

MCP	MFG
- DPP	- DPP
	- Convergence
- PDE	- PDE ?

Itô's Formula

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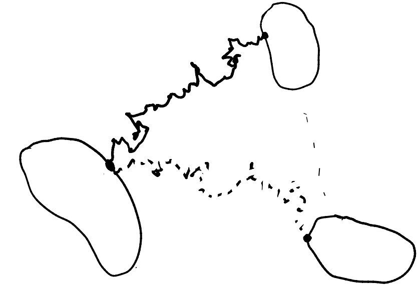


Multivariate Control Setting

[Dynamics]

$$X_s^{t,x,\alpha} = x + \int_t^s b(r, X_r^{\cdot\cdot}, \alpha_r)\, dr + \int_t^s \sigma(r, X_r^{\cdot\cdot}, \alpha_r)\, dB_r$$

[Value]

$$Y_s^{t,x,\alpha} = g(X_T^{t,x,\alpha}) + \int_s^T F(r, X_r^{\cdot\cdot}, \alpha_r, Y_r^{\cdot\cdot}, Z_r^{\cdot\cdot})\, dr - \int_s^T Z_r^{t,x,\alpha}\, dB_r \in \mathbb{R}^m$$

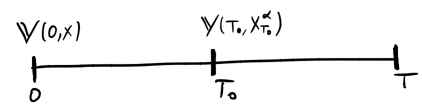
[Set Value]

$$\mathbb{V}(t,x) = \{ \mathcal{J}(t,x,\alpha) : \forall \alpha \} \; ; \quad \mathcal{J}(t,x,\alpha) \doteq Y_t^{t,x,\alpha}$$

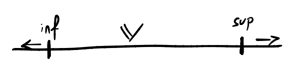
DPP

$$\mathbb{V}(0,x) = \{ \mathcal{J}(0,x,\alpha; T_0, \phi) : \forall \alpha, \phi \in \mathbb{V}(T_0, X_{T_0}^{\alpha}) \}$$

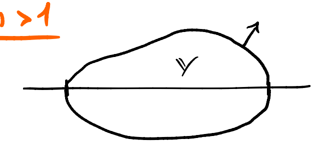
Recall: $V(0,x) = \sup_{\alpha} E\left[V(T_0, X_{T_0}^{\alpha}) + \int_0^{T_0} F \dots \right]$



m = 1



m > 1

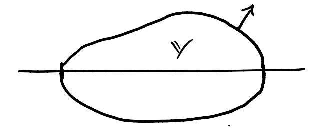


Set Value

$$\mathbb{V}(t,x) = \{ \mathcal{J}(t,x,\alpha) : \forall \alpha \}$$

m = 1

m > 1



HJB

$$\sup_{a,\xi} \; n \cdot \left[\partial_t \mathbb{V} + \frac{1}{2} \partial_{xx} \mathbb{V} + K^{\xi} + F \right](t,x,y,a) = 0 \qquad \forall (t,x,y) \in \mathcal{G}$$

Theorem (i) Suppose $\mathbb{V} \in C^{1,2}$. Then $\mathbb{V}$ solves HJB equation.

(ii) Suppose $\tilde{\mathbb{V}} \in C^{1,2}$ solves HJB and optimal $\mathcal{I}^*, \xi^*$ exists. Then $\mathbb{V} = \tilde{\mathbb{V}}$.

Moreover, $y \in \mathbb{V}_b(t,x)$ and $y = Y_t^{t,x,\alpha^*} \Rightarrow Y_s^{t,x,\alpha^*} \in \mathbb{V}_b(s, X_s^*)$

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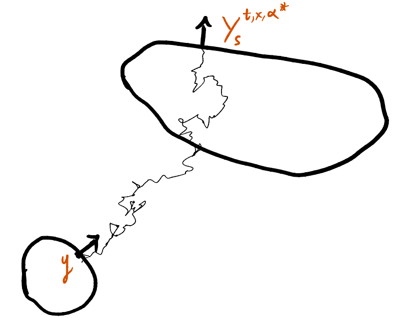


Moreover, $y \in \mathbb{V}_b(t,x)$ and $y = Y_t^{t,x,\alpha^*} \Rightarrow Y_s^{t,x,\alpha^*} \in \mathbb{V}_b(s, X_s^*)$

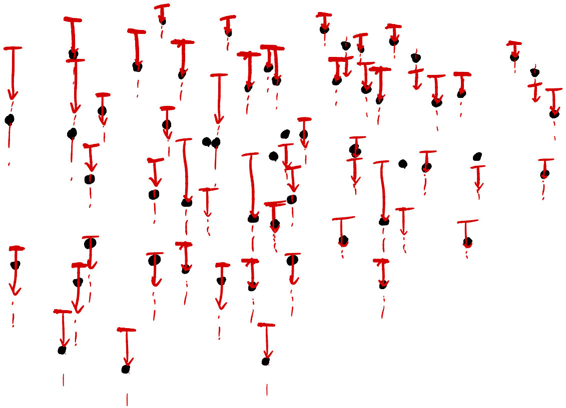
[Application] Moving Scalarization

$$\sup_{\alpha} Y_0^{0,x,\alpha} \cdot \lambda \longrightarrow \alpha^* \text{ optimal}$$

$$\alpha^* \text{ optimal} \longrightarrow \operatorname*{e.sup}_{\alpha} Y_t^{t,X_t^{\alpha^*},\alpha} \cdot n\left(t, X_t^*, Y_t^{\cdots}\right)$$

$$\alpha^* \text{ optimal} \not\longrightarrow \operatorname*{e.sup}_{\alpha} Y_t^{t,X_t^{\alpha^*},\alpha} \cdot \lambda$$

Mean-Field Games



Population

$$X_s^{t,\mu,\alpha} = X_t + \int_t^s b\left(r, X_r^{t,\mu,\alpha}, \mathcal{L}_{X_r^{t,\mu,\alpha}}, \alpha\left(r, X_r^{\cdot\cdot}, \mathcal{L}_{X_r^{\cdot\cdot}}\right)\right) dr + B_{t,s}$$

Player

$$X_s^{x,\tilde{\alpha}} = x + \int_t^s b\left(r, X_r^{x,\tilde{\alpha}}, \mathcal{L}_{X_r^{t,\mu,\alpha}}, \alpha\left(r, X_r^{x,\tilde{\alpha}}, \mathcal{L}_{X_r^{t,\mu,\alpha}}\right)\right) dr + B_{t,s}$$

Nash Equilibrium

$$\mathcal{J}(\underbrace{t,\mu,\alpha^*}_{\text{population}}; \underbrace{\alpha^*, x}_{\text{player}}) \geq \mathcal{J}(t,\mu,\alpha^*; \alpha, x)$$

$$\mathbb{V}(t,\mu) \doteq \left\{ \mathcal{J}(t,\mu,\alpha^*; \alpha^*, \cdot) \;\middle|\; \alpha^* \text{ equilibrium at } (t,\mu) \right\}.$$

Games are sensitive ..

MFG

population: $\alpha \in A$

player: $\tilde{\alpha} \in A$

$$\mathcal{J}(\alpha^*; \alpha^*) \geq \mathcal{J}(\alpha^*, \alpha)$$

α^* equilibrium

Relaxed

$\gamma \in \mathcal{P}(A)$

$\tilde{\gamma} \in \mathcal{P}(A)$

$$\mathcal{J}(\gamma^*; \gamma^*) \geq \mathcal{J}(\gamma^*, \gamma)$$

γ^* equilibrium

Global

$\Gamma \in \mathcal{P}(\alpha)$

$\alpha \in A$

$$\mathcal{J}(\Gamma; \alpha) \geq \mathcal{J}(\Gamma; \tilde{\alpha})$$

Γ equilibrium
$\alpha \in \operatorname{supp} \Gamma$

$$\mathbb{V} \neq \mathbb{V}^{\text{Relaxed}} = \mathbb{V}^{\text{Global}}$$

Games are sensitive ..

[Dynamics are state dependent]

but ..

state-dependent $\alpha(t, X_t, \mu_t)$

path-dependent $\alpha(t, X_{[0,t]}, \mu_{[0,t]})$

- $V^{state} \neq V^{path}$.

full-information $\alpha(t, X^1, \ldots, X^N)$ — weak MFE

Stability

<u>ϵ - equilibrium</u>

$$\int_{\mathbb{R}^d} \mathcal{J}(t,\mu,\alpha^*;\alpha^*,x) - \inf_{\alpha} \mathcal{J}(t,\mu,\alpha^*;\alpha,x) \, d\mu(x) \leq \epsilon$$

$$\mathbb{V}_\epsilon(t,\mu) \doteq \Big\{ \varphi \text{ Lipschitz}: \int_{\mathbb{R}^d} \big| \varphi - \mathcal{J}(t,\mu,\alpha^*;\alpha^*,\cdot) \big| \, d\mu(x) \leq \epsilon \,; \quad \alpha^* \ \epsilon\text{-equilibrium at } (t,\mu) \Big\}$$

$$\mathbb{V}(t,\mu) \doteq \bigcap_{\epsilon>0} \mathbb{V}_\epsilon(t,\mu)$$

Results.

Diffusion: state: homogeneous   discrete: state / path: homogeneous / heteregoneous

DPP

$$\mathbb{V}(0,\mu) = \left\{ \mathcal{J}\left(t,\mu,\alpha^*;\alpha^*,\cdot\,;T_0,\varphi\right) : \alpha^* \text{ equilibrium}; \quad \varphi \in \mathbb{V}(T_0,\mu_{T_0}^{\alpha^*}) \right\}$$

- Under monotonicity (uniqueness) + Itô $\Rightarrow$ Master Equation

Convergence

$$\bigcap_{\epsilon>0} \limsup_{N\to\infty} \mathbb{V}_\epsilon^{N,\mathrm{hom}}(t,\vec{x}) = \mathbb{V}(t,\mu) = \bigcap_{\epsilon>0} \mathbb{V}_\epsilon(t,\mu)$$

$(\alpha_1 = \cdots = \alpha_N)$

$$\bigcap_{\epsilon>0} \limsup_{N\to\infty} \mathbb{V}_\epsilon^{N,\mathrm{het}}(t,\vec{x}) = \mathbb{V}^{\mathrm{global}}(t,\mu) = \mathbb{V}^{\mathrm{relax}}(t,\mu)$$

$(\alpha_1, \ldots, \alpha_N)$

Thank you